Inverse Spectral Results on Toric Manifolds

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(Based on joint works with V. Guillemin and A. Uribe)

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"Interactions of symplectic and algebraic geometry"

Outline

What's next ☞

Quantization

Quantization Classical $\rightsquigarrow$ Quantum

- People would like to get a correspondence
  symplectic manifold $\rightsquigarrow$ Hilbert space
  (some) smooth functions $\rightsquigarrow$ (some) self-adjoint operators
- In literature there are many different mathematical theories towards quantization, e.g. Weyl quantization, geometric quantization, Toeplitz quantization, deformation quantization

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Semi-classical analysis Quantum $\rightsquigarrow$ classical

The guiding principle is

The Bohr Correspondence Principle

classical system = small $\hbar$ limit of quantum system

Weyl Quantization

Weyl Quantization for the model $M = T^*\mathbb{R}^n$ and $\mathcal{H} = L^2(\mathbb{R}^n)$

$$x_j \rightsquigarrow Q_j = \text{multiplication by } x_j$$
$$\xi_j \rightsquigarrow P_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$$

Heisenberg canonical commutative relation:

$$[P_j, Q_k] = \frac{\hbar}{i}\delta_{jk}.$$

There is a general rule to quantize more complicated functions, and thus get a correspondence symbols $\rightsquigarrow$ pseudodifferential operators

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Example: the energy function $\rightsquigarrow$ the Schrödinger operator:

$$H(x,\xi) = |\xi|^2 + V(x) \rightsquigarrow \hat{H} = -\hbar^2\Delta + V(x)$$

Note: Make sense for Riemannian manifolds: $T^*X \rightsquigarrow L^2(X)$

The Spectrum

The setting: Let X be a Riemannian manifold

- (quantum) $P : C^\infty(X) \to C^\infty(X)$ a semi-classical PsDO
- (classical) $p \in C^\infty(T^*X)$ the symbol of P
  - assumptions: P is self-adjoint, elliptic
  - assumptions: $p \geq 0$, $\to \infty$ if X is non-compact

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The spectrum The quantum dada are the eigenvalues of P, i.e. the λ's such that $P\varphi = \lambda\varphi$ for some $\varphi \neq 0$. They are quantum energies. They form a real and discrete sequence

$$\lambda_1(\hbar) \leq \lambda_2(\hbar) \leq \cdots \lambda_k(\hbar) \leq \cdots \to \infty$$

The main problem

The main problem that we are interested in is

The Inverse Spectral Problem

Find the relation between the symbol function $p \in C^\infty(T^*X)$ and the asymptotic behavior of the spectrum $\lambda_i(\hbar)$'s.
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An Illuminating Example: Counting Energies

Let $N(c) = \#\{i \mid \lambda_i(\hbar) < c\}$. Then according to the Weyl law,

$$N(c) \overset{\hbar \to 0}{\sim} \frac{1}{(2\pi\hbar)^n} \mathrm{Vol}(p(x,\xi) < c)$$

Note: The left hand side count the quantum energies below c, while the right hand side "count" the classical energies below c.

What's next ☞

From Smooth to Symplectic

Smooth theory

- X a smooth manifold
- G a compact Lie group
- $\tau : G \to Diff(X)$ a smooth action of G on X

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$\rightsquigarrow$

Symplectic theory

- $M = T^*X$ a symplectic manifold
- Hamiltonian G-action on M

$$g \cdot (x, \xi) = (g \cdot x, \xi \circ d\tau_{g^{-1}}) \in T^*_{g \cdot x}X$$

whose moment map $\Phi : M \to \mathfrak{g}^*$ is given by

$$\langle \Phi(x, \xi), v \rangle = \xi(v_M(x))$$

General Reduction Theory

Symplectic Reduction

Theorem (Marsden-Weinstein Theory)

For any coadjoint orbit $\mathcal{O}$ of G, G acts smoothly on $\Phi^{-1}(\mathcal{O}) \subset M$, and under suitable assumptions, $\Phi^{-1}(\mathcal{O})/G$ admits a quotient symplectic structure.

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Notion: T : Cartan subgroup of $G \rightsquigarrow$ induced T-action
$\mathfrak{t}^*_+ \subset \mathfrak{t}$: positive Weyl chamber

Assumption:

- $\alpha = \mathcal{O} \cap \mathfrak{t}^*_+$ is a regular value of Φ_T

then

$$M_\alpha := \Phi_T^{-1}(\alpha)/T = \Phi^{-1}(\mathcal{O})/G$$

Reduction of Cotangent Bundle

Consider the open subset

$$X_0 = \{x \in X \mid \mathfrak{t}_x = 0\} \subset X$$

where the induced T-action is locally free.

Fact 1: $\alpha \in \mathfrak{t}^*$ is a regular value $\iff \Phi_T^{-1}(\alpha) \subset T^*X_0$

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For simplicity assume T acts on X_0 freely. Let $B = X_0/T$.

Fact 2:

$$(T^*X)_\alpha = T^*B$$

(with a slightly different symplectic form)

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Fact 3: By choosing a connection, one can identify $\mathfrak{g}$ with the vertical tangent space of X_0 at each x

What's next ☞

Isometric Lie Group Action

General setting

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Observation: The induced linear G-action on $L^2(X)$

$$(a \cdot f)(x) = f(a^{-1}x),$$

commutes with P.

$\Longrightarrow$ For each eigenvalue λ of P, the eigenspace

$$E_\lambda = \{\varphi \in L^2(X) |\ P\varphi = \lambda\varphi\}$$

admits an induced linear action of G, i.e. each E_λ is a finitely dimensional representation of G.

Decomposition of Representations

Irreducible representation Each E_λ can be decomposed into a direct sum of irreducible representations of G:

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Isotypical representation: For each $V_\alpha \in \hat{G}$, we put all those subspaces that are isomorphic to V_α together:

$$L^2_\alpha(M) \simeq \bigoplus_\lambda n_\lambda^\alpha V_\alpha$$

Example: If $G = \mathbb{T}^n$ is the standard torus, then

$$L^2_\alpha(M) = \{f \in L^2(M) \mid \exp(v) \cdot f = e^{i\alpha(v)} f\}.$$

The Equivariant Spectrum

New decomposition and new operators So we get

- $L^2(M) = \oplus L^2_\alpha(M)$
- $P_\alpha = P|_{L^2_\alpha(M)} : L^2_\alpha(M) \to L^2_\alpha(M)$

For each $V_\alpha \in \hat{G}$, we thus get the equivariant spectrum

$$\mathrm{Spec}(P_\alpha) : \underbrace{\lambda_1(\hbar), \cdots, \lambda_1(\hbar)}_{n^\alpha_{\lambda_1}}, \underbrace{\lambda_2(\hbar), \cdots, \lambda_2(\hbar)}_{n^\alpha_{\lambda_2}}, \cdots$$

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The inverse e-spectral problem

The Inverse Spectral Problem

Can we recover p from the equivariant spectrum of P? What information of p can we get?

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The Laplace-Beltrami Operator

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- (X, g) = a Riemannian manifold without boundary
- Δ_g = the Laplace-Beltrami operator

$$\Delta_g = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i (\sqrt{|g|} g^{ij} \partial_j).$$

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$$\Delta_g = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i(\sqrt{|g|} g^{ij} \partial_j).$$

Fact: Δ_g is a second order differential operator, elliptic and self-adjoint, and its symbol is

$$\sigma(\Delta_g) = \|\xi\|^2.$$

[It is a function defined on the cotangent bundle.]

The Schrödinger Operators

The semi-classical Schrödinger operator

- $\hbar$: the semi-classical parameter (the Planck's constant)
- $V \in C^\infty(X)$: the potential function
- The Schrödinger operator

$$\widehat{H} = \hbar^2 \Delta_g + V.$$

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- $V \in C^\infty(X)$: the potential function
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$$\widehat{H} = \hbar^2 \Delta_g + V.$$

Fact: $\widehat{H}$ is a zeroth order semiclassical differential operator, elliptic and self-adjoint, and its semiclassical symbol is

$$H(x,\xi) = \|\xi\|^2 + V(x) \qquad (\textit{classical energy}).$$

[It is a function on the cotangent bundle T^*X]

The spectral problem

The spectrum We are interested the eigenvalues of $\widehat{H}$:

$$(\hbar^2 \Delta_g + V(x))\varphi = \lambda\varphi.$$

The eigenvalue λ's are real and depends on $\hbar$. [They are "energies" of quantum states.]

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Discreteness of the spectrum

Assume one of the following conditions:

- X is compact, or
- $V(x) \to +\infty$ as $x \to \infty$ if X is not compact.

then one has a discrete sequence of eigenvalues:

$$\mathrm{Spec}(\hbar^2\Delta_g + V) = \{\lambda_1(\hbar) \le \lambda_2(\hbar) \le \cdots \to \infty\}$$

The inverse spectral problem

The inverse spectral problem

The Inverse Spectral Problem

Can we determine $V(x)$ from $\mathrm{Spec}(\hbar^2\Delta_g + V)$? What can we say about the map

$$\Lambda : \mathcal{V} \to \mathbb{R}^\infty, \quad V \mapsto \{\lambda_1(\hbar), \lambda_2(\hbar), \cdots \lambda_k(\hbar), \cdots\}?$$

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Not much is known

- Weyl's asymptotic
- Gutzwiller trace formula
- (Under suitable assumptions, without semiclassical parameter $\hbar$) isospectral compactness, spectral rigidity etc

The case of $\mathbb{R}^n$

Some known results (no Lie group action) for $X = \mathbb{R}^n$:

- For $n = 1$: one can spectrally determine even potentials
- [Datchev-Hezari-Ventura 11] One can spectrally determine potentials of the form $V(x) = R(|x|)$, where R is increasing
- [Guillemin-Uribe-W 12] One can spectrally determine potentials of the form $V(x) = x^2 + \hbar^2 W$, where $W = R(|x|)$.
- [Guillemin-Uribe-W 12] For $n = 2$, one can spectrally determine potentials of the form $V(x) = x^2 + \hbar^2 W$, where W is analytic and of the form $ax_1^2 + bx_2^2 + W_4 + W_6 + \cdots$.

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Main idea:

Extra structure in the spectrum.

What's next ☞

Equivariant spectrum

Setting

- (X, g) Riemannian manifold
- G a compact Lie group acting on (X, g) isometrically
- $V \in C^\infty(M)$ is G-invariant
  - $V(x) \to \infty$ as $x \to \infty$ if X is non-compact

$\rightsquigarrow$ The G-equivariant spectrum of $\hat{H} = \hbar^2\Delta_g + V$

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New phenomena

The equivariant eigenvalues share many properties from the ordinary eigenvalues, but there are also some new phenomena. e.g.

Theorem (Abreu-Freitas)

There is no Hersch type theorem for λ_1^{inv}.

Remark: related results were proved recently [Legendre-Sena-Dias, Hall-Murphy]

Some Known Results

Known results for equivariant spectrum (with $V = 0$)

- [Brüning-Heintze, Donnelley] Equivariant heat trace asymptotic, equivariant Weyl's asymptotic
- [Guillemin-Uribe] Equivariant wave trace formula
- [Dryden-Guillemin-Sena-Dias] The equivariant spectrum of a generic toric orbifold (with toric Kähler metric) determine the toric orbifold
- [Dryden-Macedo-Sena-Dias] The equivariant spectrum of a S^1-invariant metric on S^2 determines the metric if $\ddot{g}$ is a single well

Equivariant Spectral Measure

Semiclassical parameter For simplicity let G be a torus. Then $\alpha \in \hat{G} \Rightarrow k\alpha \in \hat{G}$. Consider $\hbar$ of the form $\hbar = \frac{1}{N} (N \in \mathbb{N})$. Denote the spectrum of $\hat{H}_{\alpha/\hbar} : L^2(X)_{\alpha/\hbar} \to L^2(X)_{\alpha/\hbar}$ by

$$\lambda_1(\alpha, \hbar) \leq \lambda_2(\alpha, \hbar) \leq \lambda_3(\alpha, \hbar) \leq \cdots$$

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Equivariant spectral measure

$$\mu_{\alpha,\hbar}(f) = \text{trace } f(\hat{H}|_{L^2(X)_{\alpha/\hbar}}) = \sum_i f(\lambda_i(\alpha, \hbar)), \qquad f \in C_0^\infty(\mathbb{R})$$

Question: What is the asymptotic behavior of $\mu_{\alpha,\hbar}$ as $\hbar \to 0$? Do we get interesting spectral invariants?

Asymptotics of the Equivariant Spectral Measure

Theorem

As $\hbar \to 0$, the measure $\mu_{\alpha,\hbar}$ admits an asymptotic expansion

$$\mu_{\alpha,\hbar} \sim (2\pi\hbar)^{-\dim B} \sum_{i=0}^{\infty} \hbar^i \nu_{i,\alpha},$$

where $\nu_{i,\alpha}$'s are measures on $\mathbb{R}$ supported on the image $[c_\alpha, +\infty)$ of the reduced Hamiltonian

$$H_\alpha(y,\eta) = \langle \eta, \eta \rangle_B^*(y) + \langle \alpha, \alpha \rangle_X^*(x) + V(x).$$

In particular, the equivariant spectrum determines $c_\alpha = \min_{T^*B} H_\alpha$

- For $G = \{1\}$, see Guillemin-W 2012
- For $G = T$, see Dryden-Guillemin-Sena Dias, 2014
- For general G and P, see Guillemin-Sternberg

What's next ☞

Lagrangian Submanifolds

Basic Setting

- $\mathcal{U}_1, \mathcal{U}_2$ be open subsets of $\mathbb{R}^n$
- $\gamma : T^*\mathcal{U}_1 \to T^*\mathcal{U}_2$ a symplectomorphism
- Λ_1 a Lagrangian submanifold of $T^*\mathcal{U}_1$

$\Rightarrow$ $\Lambda_2 = \gamma(\Lambda_1)$ is a Lagrangian submanifold of $T^*\mathcal{U}_2$

$\Rightarrow$ $\Gamma = \{(x, y, -\xi, \eta) | (y, \eta) = \gamma(x, \xi)\}$ is a Lagrangian submanifold of $T^*(\mathcal{U}_1 \times \mathcal{U}_2)$

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Assumptions

- These Lagrangians are horizontal associated to $F \in C^\infty(\mathcal{U}_1)$, $G \in C^\infty(\mathcal{U}_2)$ and $W \in C^\infty(\mathcal{U}_1 \times \mathcal{U}_2)$
  - i.e. $(x, \xi) \in \Lambda_1 \Leftrightarrow \xi = \frac{\partial F}{\partial x}$ etc.
- For all y, the function $x \mapsto F(x) + W(x, y)$ has a unique critical point which is a global minimum.

Generalized Legendre Transform

Theorem (Guillmin-W 16)

Under these assumptions,

$$G(y) = \min_x (F(x) + W(x, y))$$

Note:

- The classical Legendre transform inversion formula is a special case of this theorem with $W(x, y) = -x \cdot y$.
- If we know W (thus γ), then from G (i.e. Λ_2) one can determine F (via $\Lambda_1 = \gamma^{-1}(\Lambda_2)$).

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Simple consequences

- Inverse result for $\mathbb{C}^n$: generalize Dryden-Guillemin-Sena Dias
- Get inverse results on $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$.

Case of Symplectic Toric Manifolds

Guillemin potential Now suppose X is a symplectic toric manifold. Then $B = X_0/T$ is the Delzant polytope, which is of the form

$$l_i(y) = \sum l_i^j y_j + l_i^0 \geq 0, \quad 1 \leq i \leq d.$$

Moreover,

Theorem (Guillemin)

The induced Riemannian metric on B is $\frac{1}{2}\sum_{i=1}^{d} \frac{(dl_i)^2}{l_i(y)}$

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Inverse results

Apply the generalized Legendre transform to

$$W(x,y) = \sum \frac{(dl_i(x))^2}{l_i(y)} = \sum \frac{(\sum_k l_i^k x_k)^2}{\sum_k l_i^k y_k + l_i^0}.$$

$\rightsquigarrow$ global/local inverse spectral results on general toric varieties.

Inverse Results on $\mathbb{CP}^n$

Consider the inverse problem for $\hat{H} = \hbar^2\Delta + V$ on $\mathbb{CP}^n$. Since V is $\mathbb{T}^n$-invariant, it defines a function V on P. Assume

(1) V is strictly convex in P

(2) $\frac{\partial V}{\partial y_i} < 0$ on $y_1 + \cdots + y_n = \frac{1}{2}$.

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Under these assumptions, one can determine V on the region R

$$R : y_1 > 0, \cdots, y_n > 0, \frac{1}{2} < 1 - \sum y_i < 1.$$

from the semi-classical equivariant spectrum of $\hat{H}$.

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- $\Rightarrow V(y) + W(x, y)$ admits a unique minimum

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- $\Rightarrow$ The minimum is in R

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Non-abelian Case

Multiplicity-free spaces We say (X, ω) is multiplicity free if for each α, the reduced spaces $\Phi_T^{-1}(\alpha)/T$'s is a point.

- These spaces are non-abelian versions of toric manifolds.
- e.g. generic $SU(n+1)$-coadjoint orbits as $U(n)$-manifolds

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Spectral results In this case we can still do (with modifications):

- Decomposition into irreducibles
- Spectral measure asymptotics
- Generalized Legendre transform

and thus (on going project) should get various inverse results.

What's next ☞

The Berezin-Toeplitz quantization

Symplectic $\rightsquigarrow$ Hilbert

- Classical phase space: (M, ω) = a compact Kähler manifold
  - Quantization condition: ω is integral
- (L, ∇, h) a holomorphic Hermitian line bundle over M
  - $\mathrm{curv}(\nabla) = -2\pi i \omega.$
- Quantum phase space: $\mathcal{H} = H^0(M, L)$

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Functions $\rightsquigarrow$ Operators:

- $\pi : L^2(M, L) \to H^0(M, L)$ the orthogonal projection
- $f \in C^\infty(M) \rightsquigarrow T_f : \mathcal{H} \to \mathcal{H}, s \mapsto \pi(fs)$

Problem: The space $\mathcal{H}$ is not large enough!

Semi-classical parameter

Introducing semi-classical parameter

- The semi-classical parameter $\hbar = 1/N$, where $N \in \mathbb{N}$ is large
- The quantum phase space $\mathcal{H}_N = H^0(M, L^{\otimes N})$
- The orthogonal projection $\pi_N : L^2(M, L^{\otimes N}) \to H^0(M, L^{\otimes N})$
- The Toeplitz operator $T_f^N : \mathcal{H}_N \to \mathcal{H}_N$, $s \mapsto \pi_N(fs)$

The Berezin-Toeplitz quantization:

$$f \mapsto (T_f^N)_{N\in\mathbb{N}} : \bigoplus_N \mathcal{H}_N \to \bigoplus_N \mathcal{H}_N$$

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- The semi-classical parameter $\hbar = 1/N$, where $N \in \mathbb{N}$ is large
- The quantum phase space $\mathcal{H}_N = H^0(M, L^{\otimes N})$
- The orthogonal projection $\pi_N : L^2(M, L^{\otimes N}) \to H^0(M, L^{\otimes N})$
- The Toeplitz operator $T_f^N : \mathcal{H}_N \to \mathcal{H}_N$, $s \mapsto \pi_N(fs)$

The Berezin-Toeplitz quantization:

$$f \mapsto (T_f^N)_{N\in\mathbb{N}} : \bigoplus_N \mathcal{H}_N \to \bigoplus_N \mathcal{H}_N$$

Semi-classical behavior As $N \to \infty$, one has

- $\|T_f^N\| \to \|f\|_\infty$ and $\mathrm{Tr}(T_f^N) = N^n \int_M f \frac{\omega^n}{n!} + O(N^{n-1})$
- $\|Ni[T_f^N, T_g^N] - T_{\{f,g\}}^N\| = O(1/N)$

What's next ☞

The Szegö projector

The Hardy space:

- Ω=a strictly pseudoconvex domain in a complex manifold
  - i.e. $(\sum \partial_i \bar{\partial}_j \rho) > 0$, where ρ is a defining function of Ω
- $X = \partial\Omega$ the boundary (Denote the inclusion $j : \partial\Omega \hookrightarrow \overline{\Omega}$)
  - It is a contact manifold with contact form $\alpha = j^* \mathrm{Im}(\bar{\partial}\rho)$
- $\Sigma = \{(x, \xi) \mid x \in X, \xi = r\alpha_x, r > 0)\}$ (a symplectic cone)
  - It is the characteristic variety of the C-R operator $\bar{\partial}_b$
- The Hardy space

$$H^2(X) = L^2\text{-closure of } f|_X, \text{ where } f \text{ is holomorphic in } \Omega$$

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The Szegö projector

- The Szegö proj. $=$ the orth. proj. $\Pi : L^2(X) \to H^2(X)$

Toeplitz operators a la Boutet de Monvel-Guillemin

The generalized Toeplitz operator

Toeplitz operator (Boutet de Monvel-Guillemin)

A Toeplitz operator is an operator of the form

$$Q = \Pi P \Pi,$$

where P is a classical pseudo-differential operator of order d on X (which can be chosen to commute with Π)

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The classical Toeplitz operator

- Ω=the unit disk in $\mathbb{C}$
- $H^2(X) = \mathrm{span}\{e^{i\theta} \mid i \in \mathbb{N}_{\geq 0}\}$.
- $T = T_f = \Pi M_f \Pi$, where M_f="multiplication by f"

Relation to Berezin-Toeplitz quantization

Dual circle bundle

- (L, h) the quantum line bundle of (M, ω)
- L^* the dual line bundle of L
- $\mathcal{D}$ the unit disk bundle in L^*
  - (Grauert): $\mathcal{D}$ is strictly pseudoconvex
- $\rightsquigarrow$ Can study Toeplitz operators on $X = \partial\mathcal{D}$.

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Relation

There is a canonical S^1-action on X, which preserves $H^2(X)$ and thus gives us a decomposition $H^2(X) = \bigoplus_N \mathcal{H}_N$, where

$$\mathcal{H}_N = \{f \in H^2(X) \mid f(e^{i\theta} \cdot x) = e^{iN\theta} f(x)\}$$

Fact: $\mathcal{H}_N \simeq H^0(M, L^{\otimes N})$

$\rightsquigarrow H^2(X) \simeq \bigoplus_N H^0(M, L^{\otimes N})$ and $T_f \leftrightarrow (T_f^N)$

Symbol calculus for Toeplitz operators

Symbol of Toeplitz operator $Q = \Pi P \Pi$

The symbol of Q is the restriction

$$\sigma(Q) := \sigma(P)|_{\Sigma}.$$

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Fact : The symbol is independent of the choices of P, and satisfies all nice properties that a symbol should have, e.g .

- $\sigma([Q_1, Q_2]) = \{\sigma(Q_1), \sigma(Q_2)\}$
- If Q is of order k and $\sigma(Q) = 0$, then Q is of order $k-1$.

and as like PsDO's, spectral behavior of Q is closely related to its symbol (assuming P is self-adjoint, elliptic, with positive leading symbol p so that one has discrete spectrum)

What's next ☞

Semi-classical theory via torus action

Decomposition via torus action

- Suppose X admits a $\mathbb{T}^m$-action which commutes with Π
- α an element of the weight lattice of $\mathbb{T}^m$, consider

$$H^2_\alpha(X) = L^2_\alpha(X) \cap H^2(X)$$

Let $P_{N\alpha}$ be the restriction of P on $H^2_{N\alpha}(X)$.
Want: to study the asymptotic of the spectral measure associated to the Toeplitz operators $\Pi P_{N\alpha} \Pi$.

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Fact: The $\mathbb{T}^m$-action induces a Hamiltonian $\mathbb{T}^m$-action on Σ.

- Denote the moment map by $\phi : \Sigma \to \mathfrak{t}^*$.
- Assumption: α is a regular value of ϕ, and $\mathbb{T}^m$ acts freely on $\phi^{-1}(\alpha) \rightsquigarrow$ symplectic quotient $\Sigma_\alpha = \phi^{-1}(\alpha)/\mathbb{T}^m$.

Spectral measure asymptotics for Toeplitz operators

Theorem (Guillemin-Uribe-W 17)

The spectral measure of $\Pi P_{N\alpha} \Pi$ *admits an asymptotic expansion*

$$(2\pi\hbar)^{-r} \sum_{i=0}^{\infty} \mu_{\alpha,i} \hbar^i$$

as $\hbar = 1/N \to 0$, *where* $r = \frac{1}{2} \dim \Sigma_\alpha$ *and*

$$\mu_{\alpha,0}(f) = \int_{\Sigma_\alpha} p_\alpha^* f d\sigma,$$

σ *being the symplectic volume form on* Σ_α *and* p_α *the "reduced" semi-classical symbol of* P, *i.e. the map* $p_\alpha : \Sigma_\alpha \to \mathbb{R}$ *is defined by*

$$\pi^* p_\alpha = p|_{\phi^{-1}(\alpha)},$$

where π *is the projection of* $\phi^{-1}(\alpha)$ *onto* Σ_α.

Application to toric case

Toric varieties as symplectic quotients:
Any toric variety is the reduction of $\mathbb{C}^m$ by a subtorus $G \subset \mathbb{T}^m$:

$$M = \mathbb{C}^m /\!\!/_\alpha G \qquad \text{(with } K = \mathbb{T}^m/G\text{-action)}$$

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Fact: Let $G_\alpha \subset G$ be the Lie subgroup with $\mathfrak{g}_\alpha = ann(\alpha) \subset \mathfrak{g}$. Then

- $T^*S^{2m-1} /\!\!/ G_\alpha = T^*Y$ with $Y = S^{2m-1}/G_\alpha$.
- Let $H^2(Y)$ be the G_α-invariant elements in $H^2(S^{2m-1})$. Then the $S^1 = G/G_\alpha$-action on Y induces
$$H^2(Y) = \oplus_N H^2(Y)_N.$$
  - $H^2(Y)_N$: "quantization" of $(M, N\omega)$

Theorem (Guillemin-Uribe-W 17)

Let $P : C^\infty(Y) \to C^\infty(Y)$ a K-invariant zeroth order pseudodifferential operator. Then the symbol of $Q = \Pi P \Pi$ is e-spectrally determined.

Thank you for your time!