Warwick School: Nathan PS 1 Monday, September 11, 2017.

For lecture notes, references, and links to SnapPy, SageMath, etc see:
http://dunfield.info/warwick2017

1. A 3-manifold is *irreducible* if every smoothly embedded 2-sphere bounds a 3-ball. For example, a basic topological fact is that $\mathbb{R}^3$ and S^3 are irreducible.

   (a) Show the only closed orientable 3-manifold which is prime but not irreducible is $S^2 \times S^1$.

   (b) Prove that if $\widetilde{M} \to M$ is a covering map and $\widetilde{M}$ is irreducible then so is M.

   (c) Use (b) to show that T^3 and all the lens spaces $L(p,q)$ are irreducible.

2. For an orientable closed surface S with some fixed Riemannian metric, consider the circle bundle $UT(S) = \{v \in T_* S \mid \|v\| = 1\}$ of unit-length tangent vectors. (The topology of $UT(S)$ does not depend on the metric.) Show that $UT(S)$ admits a Riemannian metric modelled on one of the eight Thurston geometries. Hint: Which geometry to pick depends on S!

3. Prove that $T^3 \# T^3$ cannot be given a geometric structure modelled on one of the eight Thurston geometries.

4. Another purely topological corollary of the Geometrization Theorem is:

   Suppose M is a closed 3-manifold. If M is not S^3 then it has a nontrivial finite cover $\widetilde{M} \to M$. Equivalently, the group $\pi_1(M)$ has a nontrivial subgroup of finite-index.

   In fact, it turns out that $\pi_1(M)$ is residually finite. Prove the corollary when M is hyperbolic. Hint: Note that $\pi_1(M)$ is a subgroup of $\mathrm{PSL}_2\mathbb{C}$ and Google "Malcev's theorem linear groups". What kind of issues would arise when tackling the general case?

5. Suppose M is a closed hyperbolic n-manifold.

   (a) Prove that every $\gamma \neq 1$ in $\pi_1(M)$ is homotopic to a unique closed geodesic.

   (b) Prove that for every $L \geq 0$ there are finitely many closed geodesics of length at most L.

   (c) Prove that the isometry group of M is finite.

6. The remaining problems all involve practical computation with hyperbolic structures, so the first step is to download and install **SnapPy 2.5.4** from http://snappy.computop.org

7. (a) Load the manifold $v1234$ and name it V.

   (b) Use the browser to find the volume, Dirichlet domain, and symmetry group of V.

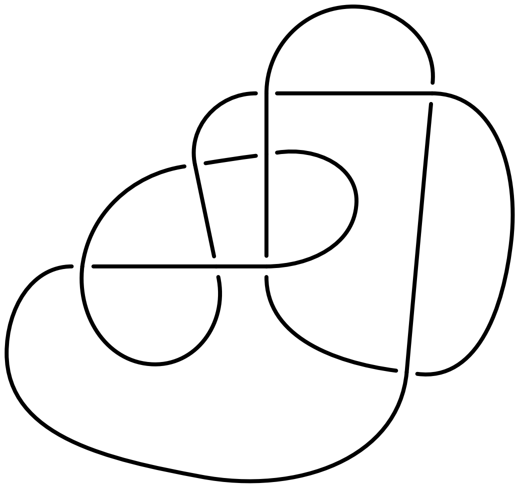
   (c) Like any manifold in SnapPy, the object V is really a particular *triangulation* of this hyperbolic manifold. Back at the command line, determine the number of tetrahedra in the triangulation V. Hint: Use tab completion by typing `V.<tab-key>`.

   (d) The manifold V has one cusp. Back the browser, do Dehn filling along the meridian curve. What closed manifold do you get?

8.

(a) Use SnapPy to find the name in the Rolfsen table for the link shown at right.

(b) Is the projection at right the same as the one that's stored in SnapPy?



9. In my lecture, I mostly focused on manifolds with cusps, but SnapPy also works with closed manifolds. In particular, it comes with the Hodgson-Weeks census of small-volume closed hyperbolic 3-manifolds, which is called `OrientableClosedCensus`.

(a) Use the "?" operator to find out more about `OrientableClosedCensus`; in particular, how many manifolds are in it?

Also, interrogate the orientable *cusped* census to get ideas on how to select various types of manifolds for the later parts of this question.

(b) Closed manifold in SnapPy are represented as Dehn fillings on cusped manifolds. You can do Dehn filling in the browser, via the `dehn_fill` method, or as part of the specification that you given to `Manifold`. For example, typing `A = Manifold('4_1(1,2)')` gives the closed 3-manifold which is $\frac{1}{2}$-Dehn surgery on the figure 8 knot. Use the method `is_isometric_to` to show that A is the sixth manifold the `OrientableClosedCensus`. Warning: In Python, lists are numbered starting from 0 rather than 1.

(c) Find the unique manifold M in the original `OrientableClosedCensus` whose volume is between 3.0 and 3.1 and whose first homology is $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

(d) Find a description of M as Dehn surgery on a 3-component link in S^3. Hint: Unfill the cusp in the default description of M and then drill out the shortest geodesic twice.