$\mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$

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These exercises were designed to be part of a mini-course given in Morelia (Mexico) but it should be possible to do them with some knowledge on elementary group theory and integration. The authors will be happy to receive any question, comment or interesting solutions to their exercises.

Note: we identify $\mathbb{C}$ to $\mathbb{R}^2$, complex numbers to vectors in the plane, real and imaginary parts to horizontal and vertical components.

In Section 1 we review the construction of the Haar measure on $\mathrm{SL}(2, \mathbb{R})$. Section 2 presents a section of the geodesic flow acting on the space of translation tori. This section allows, by integrating the return time on the section, to recover the volume of this space. A similar calculation can be made by constructing a section for the horocyclic flow, for which the first return map is called the BCZ map (see [AC13]).

References: [Arn94] (the geodesic flow on the modular surface), [Via] (general reference for translation surfaces), [Gar] (volumes of $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SL}(n, \mathbb{Z})$ by different methods).

1 The Haar measure on $\mathrm{SL}(2, \mathbb{R})$

Question 1 (Matrix spaces and matrix groups)**.** The space of 2×2 matrices with elements in $\mathbb{R}$, denoted by $\mathrm{Mat}_2(\mathbb{R})$, is isomorphic to $\mathbb{R}^4$ (vector space, scalar product, Lebesgue measure, etc.). Left and right multiplication define a left and a right action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{Mat}_2(\mathbb{R})$, both preserving $\mathrm{SL}(2, \mathbb{R})$. Recall that $\mathrm{SL}(2, \mathbb{R})$ is a smooth hypersurface of $\mathrm{Mat}_2(\mathbb{R})$ and that $\mathrm{GL}_2^+(\mathbb{R}) \simeq \mathbb{R}_+^* \times \mathrm{SL}(2, \mathbb{R})$ (Lie group isomorphism).

A natural way to define a measure on $\mathrm{SL}(2, \mathbb{R})$ is therefore to consider the cone measure, where measurable subsets of $\mathrm{SL}(2, \mathbb{R})$ are intersections of $\mathrm{SL}(2, \mathbb{R})$ with measurable subsets of $\mathrm{Mat}_2(\mathbb{R})$, and the measure of such a subset A is the Lebesgue measure in $\mathrm{Mat}_2(\mathbb{R})$ of the cone from 0 to A, ie, of the set $\{ \lambda x, \ \ \lambda \in [0, 1], \ \ x \in A \}$.

1. Show that the Lebesgue measure on $\mathrm{Mat}_2(\mathbb{R})$ is left- and right-invariant by $\mathrm{SL}(2, \mathbb{R})$. Show that the cone measure on $\mathrm{SL}(2, \mathbb{R})$ is left- and right-invariant by $\mathrm{SL}(2, \mathbb{R})$. Since $\mathrm{SL}(2, \mathbb{R})$ is a Lie group, a left-invariant measure is unique up to normalization, and is called a left Haar measure. On $\mathrm{SL}(2, \mathbb{R})$, the left and right Haar measures coincide.

Let $K = \mathrm{SO}(2)$, let A be the set of diagonal matrices with positive diagonal entries and determinant 1, and let N be the group of upper triangular matrices with diagonal elements equal to 1. All these groups are one-parameter subgroups of $\mathrm{SL}(2, \mathbb{R})$ and we choose the following parametrization.

$$r_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad a_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \quad \text{and} \quad n_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}.$$

These groups are used to define two decompositions of matrices in $\mathrm{SL}(2, \mathbb{R})$. These decompositions can be used to define coordinates for $\mathrm{SL}(2, \mathbb{R})$. We will use these coordinates to compute explicit densities for the Haar measure on $\mathrm{SL}(2, \mathbb{R})$.

Question 2 (KAN and KAK decomposition)**.**
1. Using the Gram-Schmidt orthonormalization process, prove that any matrix in $\mathrm{SL}(2, \mathbb{R})$ decomposes as a product kan where $(k, a, n) \in K \times A \times N$.
2. Prove that $K \times A \times N \to \mathrm{SL}(2, \mathbb{R})$ is a homeomorphism. Deduce that $\pi_1(\mathrm{SL}(2, \mathbb{R}))$ is $\mathbb{Z}$.
3. Show that any matrix $m \in \mathrm{SL}(2, \mathbb{R})$ decomposes as $k_1 a k_2$ where $(k_1, a, k_2) \in K \times A \times K$ (hint: use the singular decomposition of ${}^t m\, m$ and consider a square root).
4. Prove that $K \times A \times K \to \mathrm{SL}(2, \mathbb{R})$ is 1-to-1 except on the preimage of id.

To compute the density of the Haar measure, we will find the height of an infinitesimal cone from the origin to a given matrix $m \in \mathrm{SL}(2, \mathbb{R})$ and then compute a determinant. By height of the cone, we mean the distance from the origin to the tangent space to $\mathrm{SL}(2, \mathbb{R})$ at m.

Question 3 (The Haar measure in coordinates KAN and KAK). 1. Viewing $\mathrm{SL}(2,\mathbb{R})$ as a hypersurface in $\mathrm{Mat}_2(\mathbb{R})$, prove that the normal vector to $\mathrm{SL}(2,\mathbb{R})$ at a given point $m \in \mathrm{SL}(2,\mathbb{R})$ is ${}^t m^{-1}$.

2. Compute the height of an infinitesimal cone from the origin at $m \in \mathrm{SL}(2,\mathbb{R})$.
3. Let $m(\theta, t, s) = r_\theta a_t n_s$. Compute the three derivatives $v_\theta = \frac{d}{d\theta} m$, $v_t = \frac{d}{dt} m$ and $v_s = \frac{d}{ds} m$ and show that all three are of the form $r_\theta m'$ where m' is an element of the Lie algebra $(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$, which is the tangent space at identity to $\mathrm{SL}(2,\mathbb{R})$, ie the set of matrices with trace zero. Hint: differentiation commutes with product by a constant (including product by a constant matrix).
4. Show that the normal vector n at m may also be written in the form $r_\theta n'$.
5. Compute the density of the Haar measure on $\mathrm{SL}(2,\mathbb{R})$ in the coordinates given by the KAN decomposition.
6. Using the same method, compute the density of the Haar measure on $\mathrm{SL}(2,\mathbb{R})$ in the coordinates given by the KAK decomposition.

Question 4. Compute the volume in $\mathrm{SL}(2,\mathbb{R})$ of the ball

$$B(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R}), \quad a^2+b^2+c^2+d^2 < R^2 \right\}$$

where R is a positive real.

Hint: Use the KAK decomposition of $\mathrm{SL}(2,\mathbb{R})$.

2 The space of unit-area translation tori

To a matrix A in $\mathrm{SL}(2,\mathbb{R})$ we can associate the lattice Λ in $\mathbb{C}$ generated by the two columns of A. This lattice is unimodular, ie has co-area one, ie the quotient torus $\mathbb{C}/\Lambda$ has area one, ie any basis of Λ spans a parallelogram of area one. When needed we denote Λ by $\Lambda(A)$.

Question 5. 1. Let $A \in \mathrm{SL}(2,\mathbb{R})$ and $B \in \mathrm{SL}(2,\mathbb{Z})$. Show that $\Lambda(AB) = \Lambda(A)$.

2. Given $A \in \mathrm{SL}(2,\mathbb{R})$, show that the set $\{ B \in \mathrm{SL}(2,\mathbb{R}), \ \Lambda(AB) = \Lambda(A) \}$ is exactly $\mathrm{SL}(2,\mathbb{Z})$.
3. Conclude that $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ parametrizes the space of unimodular lattices.

The torus $\mathbb{C}/\Lambda$ naturally carries a translation structure, ie an atlas to $\mathbb{R}^2$ whose transition maps are translations, where coordinate charts are local inverses of the projection map $\mathbb{C} \to \mathbb{C}/\Lambda$. Actually any unit-area translation torus is some $\mathbb{C}/\Lambda$, so their moduli space is the same as that of unimodular lattices, ie $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ by the previous question.

Define the systole of a lattice Λ as

$$\mathrm{sys}(\Lambda) = \min_{v \in \Lambda \setminus \{0\}} \|v\|.$$

In the following we will use the following version of the diagonal flow: $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} = a_{2t}$.

Question 6. 1. Show that for any $\epsilon > 0$ the set $\{\Lambda; \mathrm{sys}(\Lambda) \geq \epsilon\}$ is a compact set. Hint: show that $\{\Lambda; \mathrm{sys}(\Lambda) \geq \epsilon\}$ is contained in $K(g_t)_{t \in [-\log(\epsilon), \log(\epsilon)]}(n_s)_{s \in [0,1]}/\mathrm{SL}(2,\mathbb{Z})$ for $\epsilon < 1$.

2. Show that $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ is not compact.

Question 7. Introduce $B_- = \{ v \in \mathbb{C}, \ 0 < \operatorname{Re} v < 1, \ \operatorname{Im} v < 0 \}$ and $B_+ = \{ v \in \mathbb{C}, \ 0 < \operatorname{Re} v < 1, \ \operatorname{Im} v > 0 \}$.

Here we consider a unimodular lattice Λ in $\mathbb{C}$, with no horizontal or vertical vector.

1. Show that the set $\Omega \subset \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of such lattices is of full measure in the space of unimodular lattices.
2. Show that there exists a unique pair (v_-, v_+) of nonzero vectors in Λ satisfying $v_- \in B_-$, $v_+ \in B_+$, $\operatorname{Im} v_- = \max_{B_-} \operatorname{Im}$, $\operatorname{Im} v_+ = \min_{B_+} \operatorname{Im}$.
3. Show that (v_-, v_+) is an oriented basis of Λ.
4. Show $\operatorname{Re}(v_- + v_+) \geq 1$.
   The parallelogram spanned by v_- and v_+ is called the Veech parallelogram of the torus $\mathbb{C}/\Lambda$.
5. Let $D = \{ (v_-, v_+) \in B_- \times B_+, \ \det(v_-, v_+) = 1 \}$, which can be seen as a subset of $\mathrm{SL}(2,\mathbb{R})$.
   Show that the closure of D is a fundamental domain for $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, ie, any class of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ has at least one element in it, and if it has two, they are in the boundary.

Define the vertical translation flow ϕ_t on $\mathbb{C}/\Lambda$ by $\phi_t(v) = v + i\,t$ (modulo Λ).

Question 8. Consider $(v_-, v_+) \in D$ and let $\Lambda = \mathbb{Z}v_- + \mathbb{Z}v_+$. Define I_Λ as the interval $[0, \operatorname{Re}(v_- + v_+)) \subset \mathbb{C}$.

1. Show that I_Λ embeds into $\mathbb{C}/\Lambda$ by the natural projection $\mathbb{C} \to \mathbb{C}/\Lambda$.
2. Let $T_\Lambda : I_\Lambda \to I_\Lambda$ be the first return time of the vertical flow to I_Λ.
   Prove that it is an interval exchange map on two intervals. What are their lengths?
3. Understand that an exchange of two intervals is semi-conjugate to a rotation (or translation) on the circle (or one-dimensional torus) $\mathbb{R}/\operatorname{Re}(v_- + v_+)\mathbb{Z}$. What is the angle of rotation (or length of translation)?

Question 9. 1. Show that $\mu(D) = \pi^2/12$ where μ is the Haar measure on $\mathrm{SL}(2,\mathbb{R})$.
2. Deduce that the space of lattices carries an $\mathrm{SL}(2,\mathbb{R})$-invariant measure of total mass $\pi^2/12$.

The flow g_t acting on the space of lattices $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ is called the Teichmüller geodesic flow. Recall from the previous section that $D = \{\,(v_-, v_+) \in B_- \times B_+,\ \det(v_-, v_+) = 1\,\}$ is a fundamental domain in $\mathrm{SL}(2,\mathbb{R})$ for the action of $\mathrm{SL}(2, Z)$. Therefore its volume gives the measure of the space of lattices.

We say that a Teichmüller geodesic $(g_t\Lambda)_{t\in\mathbb{R}}$ is forward divergent (respectively backward divergent) if $\mathrm{sys}(g_t\Lambda) \to 0$ as $t \to +\infty$ (respectively $\mathrm{sys}(g_t\Lambda) \to 0$ as $t \to -\infty$).

Question 10. Show that the Teichmüller geodesic from Λ is forward divergent (respectively backward divergent) if and only if Λ has a vertical saddle connection (respectively horizontal saddle connection).

Hint: show that if Λ has a saddle connection of the form $(\epsilon, \pm\epsilon)$, then $g_t\Lambda$ has no short saddle connection for t in an appropriate range.

Define $S = \{\,(v_-, v_+) \in D,\ \ Re(v_- + v_+) = 1\,\}$ and, for $(v_-, v_+) \in D$, the Rauzy time

$$t_R(v_-, v_+) = -\log(\max(\operatorname{Re}(v_-), \operatorname{Re}(v_+))).$$

Question 11. 1. Show that for any Λ in D, $g_{t_R(\Lambda)}\Lambda \in S$.
2. Deduce that S is a section for g_t that misses only divergent Teichmüller geodesics, ie if Λ has no horizontal nor vertical saddle connection then its Teichmüller geodesic crosses S infinitely often.

We consider the map $\widehat{R} : S \to S$ defined as the first return map of the Teichmüller flow to S.

Question 12. 1. Show that

$$\widehat{R}(v_-, v_+) = \begin{cases} (v_- - v_+, v_-) & \text{if } \operatorname{Re}(v_-) < \operatorname{Re}(v_+) \\ (v_-, v_+ - v_-) & \text{if } \operatorname{Re}(v_-) > \operatorname{Re}(v_+) \end{cases}.$$

   In particular, the image of $\widehat{R}$ only depends on the real parts of v_- and v_+.
2. Show that is $\widehat{R}$ is invertible and that the image of $\widehat{R}^{-1}$ only depends on $\operatorname{Im}(v_- + v_+)$.

The Farey map $F : [0,1] \to [0,1]$ and the Gauss map $G : [0,1] \to [0,1]$ are defined as follows

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } x < 1/2 \\ \frac{1-x}{x} & \text{if } x \geq 1/2 \end{cases} \qquad G(x) = \left\{\frac{1}{x}\right\}.$$

where $\{\alpha\}$ denotes the fractional part of the number α.

Question 13. Let $(v_-, v_+) \in S$, and let Λ be the corresponding lattice and $T_\Lambda : I_\Lambda \to I_\Lambda$ the associated interval exchange map on two intervals, defined in a previous question.

In this question we explore the link between an induced map of T_Λ on a subinterval and the map $T_{\Lambda'} : I_{\Lambda'} \to I_{\Lambda'}$ where Λ' is the lattice generated by $(v'_-, v'_+) = \widehat{R}(v_-, v_+)$.

Notation: $(\lambda_+, \lambda_-) = (\operatorname{Re} v_+, \operatorname{Re} v_-)$; $(\lambda'_+, \lambda'_-) = (\operatorname{Re} v'_+, \operatorname{Re} v'_-)$.
1. Show that the lengths of the 2 subintervals of I_Λ exchanged by T_Λ are λ_+, λ_-.
2. Let $I'_\Lambda = [0, \max(\lambda_+, \lambda_-)$. Consider the map T'_Λ induced from T_Λ on the subinterval I'_Λ, ie the first return map to I'_Λ. Show it is an interval exchange map on two intervals. What are their lengths?
3. Show that $T_{\Lambda'} : I_{\Lambda'} \to I_{\Lambda'}$ is conjugate to $T'_\Lambda : I'_\Lambda \to I'_\Lambda$ by a rescaling taking the total length to be 1.
4. Establish a link between the Farey map and the map taking the old lengths to the new lengths.
5. Denoting by R the map sending (λ_+, λ_-) to (λ'_+, λ'_-), notice that it can be one of two linear maps.
6. What is the link between R and $\widehat{R}$?

Going from T_Λ to T'_Λ is called Rauzy induction on interval exchange maps; they are first return maps to two different intervals for the vertical flow on the same torus. Going from T_Λ to $T_{\Lambda'}$ is the renormalized Rauzy induction.

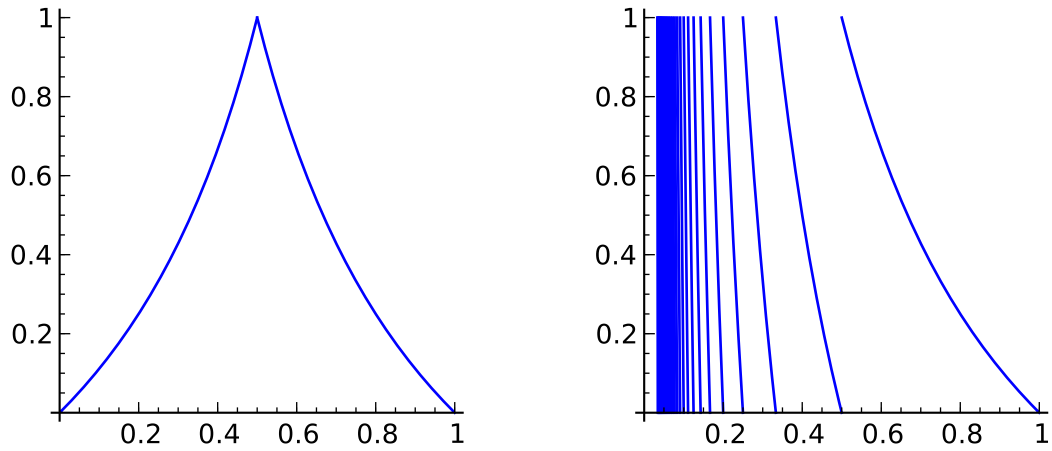

Figure 1: The Farey and the Gauss map. Notice that their rightmost branch coincides!

Question 14 (An acceleration)**.** Let

$$S' = \{\, (v_-, v_+) \in S,\ \ (\mathrm{Re}(v_-) < \mathrm{Re}(v_+) \text{ and } \mathrm{Im}(v_- + v_+) < 0) \text{ or } ((\mathrm{Re}(v_-) > \mathrm{Re}(v_+) \text{ and } \mathrm{Im}(v_- + v_+) > 0) \,\} \,.$$

1. Show that S' is a section for the geodesic flow, compute the first return map $\widehat{Z}$ and the return time t_Z. Hint: since $S' \subset S$, at each point in S' the first return to S' is some iterate of the first return to S.
2. Show that $\widehat{Z}$ only depends on the real parts of v_- and v_+ and call Z the corresponding map on the real parts. Find a link with the Gauss map.

This is the Zorich acceleration of the Rauzy–Veech induction.

Question 15 (Invariant measures)**.** Define a measure ν_R on S by

$$\nu_{\widehat{R}}(A) = \nu(\{\, (x,t),\ \ x \in A,\ t \in [0, t_R(x))\,\})$$

where ν is the Haar measure on $\mathrm{SL}(2,\mathbb{R})$. Define $\nu_{\widehat{Z}}$ similarly on S'.

1. Show $\nu_{\widehat{R}}$ and $\nu_{\widehat{Z}}$ are invariant for $\widehat{R}$ and $\widehat{Z}$ respectively. Hint: no computation; only use g_t-invariance of ν.
2. Define ν_R on the set of pairs (λ_+, λ_-) with sum 1 as

$$\nu_R(A) = \nu(\{\, (v_-, v_+) \in S,\ \ (\mathrm{Re}(v_+), \mathrm{Re}(v_-)) \in A\,\})$$

   and define ν_Z similarly using S'. Show that ν_R and ν_Z are invariant measures for R and Z respectively.
3. Find invariant measures for F and G, absolutely continuous with respect to Lebesgue measure.
4. Show that the invariant measure for F has infinite mass while the invariant measure for G has finite mass.

This last fact explains the interest of Zorich induction over Rauzy induction.

References

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