CHEAT SHEET

SOME USEFUL BACKGROUND MATERIAL

Unique perpendiculars. Given any two disjoint geodesics in the hyperbolic plane $\mathbb{H}^2$ with distinct endpoints at infinity, there is a unique geodesic that is perpendicular to both.

Isometries of the hyperbolic plane. The group of orientation-preserving isometries of $\mathbb{H}^2$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. In the upper half-plane model for $\mathbb{H}$ we can think of an element of $\mathrm{PSL}(2,\mathbb{R})$ as a Möbius transformation $z \mapsto \dfrac{az+b}{cz+d}$. We have the following classification of isometries:

(1) **Elliptic.** An element of $\mathrm{Isom}^+(\mathbb{H}^2)$ is *elliptic* if it fixes a point in $\mathbb{H}^2$. In this case the map will be a rotation about the fixed point, and the map has no fixed points on $\partial\mathbb{H}^2$. In $\mathrm{PSL}(2,\mathbb{R})$, the absolute value of the trace of an elliptic element is strictly less than 2.

(2) **Parabolic.** An element of $\mathrm{Isom}^+(\mathbb{H}^2)$ is *parabolic* if it fixes exactly one point in $\partial\mathbb{H}^2$. In this case the map is conjugate to $z \mapsto z+1$ in the upper half-plane model. In $\mathrm{PSL}(2,\mathbb{R})$, the absolute value of the trace of a parabolic element is precisely 2. (Note: we do not count the identity map as parabolic.)

(3) **Hyperbolic.** An element of $\mathrm{Isom}^+(\mathbb{H}^2)$ is *hyperbolic* if it fixes two points in $\partial\mathbb{H}^2$. In this case the map is a translation along an invariant geodesic, so we can think of one of the two fixed points on $\partial\mathbb{H}^2$ as an attracting fixed point and the other as a repelling fixed point. In $\mathrm{PSL}(2,\mathbb{R})$, the absolute value of the trace of a hyperbolic element is strictly greater than 2. Hyperbolic isometries of $\mathbb{H}^2$ are sometimes called *loxodromic*.

Hyperbolic surface. A surface S *admits a hyperbolic metric* if there exists a complete, finite-area Riemannian metric on S of constant curvature -1 and if ∂S is totally geodesic, that is, geodesics in ∂S are geodesics in S. A surface endowed with a hyperbolic metric is a *hyperbolic surface*. Any surface with negative Euler characteristic admits a hyperbolic metric.

Alternatively, we can define a (closed) hyperbolic surface S in terms of charts. Then we say that a closed surface S is hyperbolic if there is an open cover (U_i) of S with maps $\phi_i : U_i \to \mathbb{H}^2$ where each ϕ_i is an orientation-preserving homeomorphism onto its image, and satisfying the condition that whenever U_i and U_j have nonempty intersection, then the restriction of $\phi_j \circ \phi_i^{-1}$ to each connected component of $\phi_i(U_i \cap U_j)$ is the restriction of an orientation-preserving isometry of $\mathbb{H}^2$.

Recall that a *Riemannian metric* is a smoothly varying choice of inner product on the tangent space of a smooth manifold. On a surface, we can specify a Riemannian metric on an open set $U \subset \mathbb{R}^2$ via four smooth functions $g_{ij} : U \to \mathbb{R}$ satisfying two properties:

(1) $g_{12}(p) = g_{21}(p)$ for all $p \in U$, and
(2) for all $p \in U$, the matrix $\begin{bmatrix} g_{11}(p) & g_{12}(p) \\ g_{21}(p) & g_{22}(p) \end{bmatrix}$ is positive definite.

The universal cover of any closed hyperbolic surface S_g is isometric to $\mathbb{H}^2$, and hence any hyperbolic surface S_g is isometric to the quotient of $\mathbb{H}^2$ by a free, properly discontinuous action by isometries of $\pi_1(S_g)$. A deck transformation corresponding corresponding to a nontrivial element of $\pi_1(S_g)$ that is not homotopic into a puncture is a hyperbolic isometry of $\mathbb{H}^2$.

One can also construct a surface of genus $g \geq 2$ with a hyperbolic metric by taking a geodesic $4g$-gon in $\mathbb{H}^2$ with interior angle sum 2π and then identifying opposite sides.

Geodesic representatives of curves. We will rely heavily on the following important property of hyperbolic surfaces: if α is a closed curve in a hyperbolic surface S (and α is not homotopic to a puncture on S), then there is a unique geodesic closed curve γ in the homotopy class of α. This can be proved by examining the lift of α to the universal cover of S.

Hyperbolic structure. A *hyperbolic structure* on a surface S is a diffeomorphism $\phi : S \to X$, where X is a hyperbolic surface. The map ϕ is called the *marking*, and we will use the notation (X, ϕ), or simply X, to denote the *marked hyperbolic surface*.

Riemann surface. A *Riemann surface* X is a 1-dimensional complex manifold. Another way to say this is that there is an atlas of charts, called *coordinate maps*:

$$(U_\alpha,\ \phi_\alpha : U_\alpha \to \mathbb{C})$$

where the *transition maps* $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}$ are biholomorphic. Two Riemann surfaces are *isomorphic* if there is a biholomorphic homeomorphism from one to the other.

It is useful to note that, if S_g is a closed surface of genus $g \geq 2$, then there is a bijective correspondence between the set of isomorphism classes of Riemann surfaces homeomorphic to S_g, and the set of isometry classes of hyperbolic surfaces homeomorphic to S_g.

Conformal maps. A map $f : X \to Y$ between two Riemann surfaces X, Y is a *conformal map* if it is bijective and holomorphic. Note that conformal maps also have holomorphic inverses.

Complex dilatation. Let U, V be open subsets of $\mathbb{C}$, and let $f : U \to V$ be a homeomorphism that is smooth except at a finite number of points. Let $p \in U$ be a point where f is differentiable. Then the *complex dilatation* of f is given by $\mu_f = f_{\bar{z}}/f_z$; recall that f is holomorphic if and only if $\mu_f \equiv 1$ and that f is orientation-preserving if and only if $|\mu_f| < 1$.

Other kinds of dilatation. Let $f : U \to V$ and $p \in U$ be as above, and assume further that f is orientation-preserving. The *dilatation of f at p* is given by:

$$K_f(p) = \frac{|f_z(p)| + |f_{\bar{z}}(p)|}{|f_z(p)| - |f_{\bar{z}}(p)|} = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}.$$

The *dilatation* of the map f is given by

$$K_f = sup\{K_f(p)\}$$

where we take the supremum over all points p at which f is differentiable. Note that K_f satisfies $1 \leq K_f \leq \infty$.

Quasi-conformal maps. A map f as above is *quasiconformal*, or K_f*-quasiconformal*, if $K_f < \infty$. It is a relatively straightforward exercise to show that if $f : X \to Y$ is a homeomorphism of Riemann surfaces, then $K_f = 1$ if and only if f is conformal.

Holomorphic quadratic differentials.. A holomorphic quadratic differential on a Riemann surface X with atlas $\{z_i \mid U_i \to \mathbb{C}\}$ is given by $\{\phi_i(z_i)dz_i^2\}$ such that

(1) each map ϕ_i is holomorphic with finitely many zeros;

(2) the atlas is invariant under change of local coordinates, that is, $\phi_j(z_j)\left(\frac{dz_i}{dz_j}\right)^2 = \phi_i(z_i)$ for any two coordinate charts z_i, z_j.

The set of all holomorphic quadratic differentials $\mathrm{QD}(X)$ on a Riemann surface X is a vector space over $\mathbb{C}$.