TEICHMÜLLER SPACE AND MAPPING CLASS GROUPS

SOME EXERCISES

Key exercise from hyperbolic geometry.

(1) Show that the length of three alternating sides determines a right-angled hexagon up to orientation-preserving isometry.

Mapping class group basics.

(2) Prove **Alexander's Lemma**, which states that the mapping class group of a disk is trivial.

(3) Find a counterexample to the following, which could be viewed as a (false) generalisation of Alexander's Lemma to higher genus surfaces: if C is a collection of simple closed curves on a closed surface S_g with the property that $S \setminus C$ is a collection of disks, and if f is a mapping class that fixes each curve in C, then f is the identity in $\mathrm{Mod}(S_g)$. (A slightly harder exercise is to find a counterexample to the following: if f fixes every isotopy class of curves in a surface, then f must be the identity in the mapping class group of that surface.)

(4) Find examples of mapping classes that are both periodic and reducible.

Teich(S).

(5) Prove that length functions are continuous. More precisely, show that if S is any hyperbolic surface and if c is a fixed isotopy class of simple closed curves in S, then the function $\mathrm{Teich}(S) \to \mathbb{R}$ given by $\mathcal{X} \mapsto \ell_{\mathcal{X}}(c)$ is continuous.

(6) Recover the bijective correspondence between $\mathrm{Teich}(T^2)$ and $\mathbb{H}^2$ using the set of orientation-preserving maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, up to rotations and dilations, as an intermediary.

(7) In the map from $\mathrm{Teich}(S)$ to $\mathrm{DF}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\,\mathrm{PGL}(2,\mathbb{R})$, identify all four choices made here and explain why they don't matter.

(8) In the notation of the proof of the $(9g-9)$-Theorem, show carefully that $A(s+2\pi) = B(s)$. Do the two remaining cases: $s = 2\pi$ and $s > 2\pi$.

Measured foliations.

(9) Sketch the measured foliations corresponding to the following (multi-)curves on the surface S_2:
  (a) a separating curve;
  (b) a separating curve together with a (disjoint) nonseparating curve.

(10) Show that the following two spines of a pair of pants give rise to equivalent foliations:
  (a) a boundary component b together with a separating arc joining b to itself, where the other two boundary components lie on different sides of b;
  (b) two boundary components joined by a nonseparating arc.

(11) Let p be a singular point in a measured foliation, and let m_p denote the number of prongs in the singularity at p. Verify directly (in other words, without citing the Euler-Poincaré formula) that the sum $(2 - m_p)$ taken over all singularities p in a measured foliation is preserved under Whitehead moves.

(12) Explain carefully why a quadratic differential with local expression $q(z) = z^k dz^2$ corresponds to a $(k+2)$-prong singularity in the corresponding measured foliation. One starting point is to consider the case $k = 0$ and consider when a complex number α satisfies $\alpha^2 > 0$ and $\alpha^2 < 0$.

Miscellaneous.

(13) Show that the set of all $(m, s, t) \in \mathbb{R}^3_{\geq 0}$ satisfying a degenerate triangle inequality forms a cone homeomorphic to $\mathbb{R}^2$.

(14) Consider a hyperelliptic involution ι of the surface S_g. Show that the canonical reduction system for ι is empty.

(15) There is a natural map from $\mathrm{Mod}(S_g)$ to the group of simplicial automorphisms of the curve graph. Show that this map is injective using the action of $\mathrm{Mod}(S_g)$ on $\mathrm{PMF}(S_g)$.