# ON THE GEOMETRY OF OUTER SPACE - PROBLEM SET 1

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(1) Prove that the two marked metric graphs in Example 1 describe the same point in Outer Space.

(2) Prove that the action of $\mathrm{Out}(F_n)$ on Outer Space as defined in Lecture 1 is well defined:

  (a) If $(\Gamma, m, \ell), (\Gamma', m', \ell')$ are equivalent and $\Phi \in \mathrm{Aut}(F_n)$ then their images $(\Gamma, m, \ell)\Phi, (\Gamma', m', \ell')\Phi$ are equivalent.

  (b) Prove that the action of $\mathrm{Inn}(F_n)$ on Outer Space is trivial.

(3) Let $\{a, b, c\}$ be a basis of $F_3$, let $R_3$ be the rose with 3 pettals which are labeled by this basis. Consider the automorphism $\Phi(a) = ab, \Phi(b) = bab, \Phi(c) = ca$. Let $x$ be the marked graph $(R, \mathrm{id})$ with edge lengths $\frac{1}{6}, \frac{1}{5}, \frac{19}{30}$ and let $y$ be the marked graph $(R, \Phi)$ with edge lengths $\frac{1}{5}, \frac{1}{20}, \frac{3}{4}$. Construct a path in Outer Space between these points.

(4) Compute the maximal dimension and minimal dimension of a simplex in Outer Space.

(5) Show that Outer Space is a locally finite simplicial complex.

(6) Show that the stabilizer of a point in Outer Space is finite.

(7) Prove that if $d(x, y) = 0$ then $x = y$.

(8) Prove that if $h\colon \Gamma \to \Gamma'$ is a homotopy equivalence which is locally injective then $h$ is a homeomorphism. (Hint: try to extend $h$ to a convering of $\Gamma'$).

(9) Prove that reduced Outer Space $\mathcal{X}_n^R$ equivariantly deformation retracts to $\mathcal{X}_n$.

(10) Let $K_n$ be the simplicial complex whose simplicies corresond to marked $F_n$-graphs, with the equivalence relation $(\Gamma, m) \sim (\Gamma', m')$ if there exists a homeomorphism $h\colon \Gamma \to \Gamma'$ so that $m'$ is freely homotopic to $m$. Faces of a simplex correspond to forest collapse. There is an $\mathrm{Out}(F_n)$ action on $K_n$: $(\Gamma, m)\phi = (\Gamma, m \circ \phi)$. Prove that there is an equivariant deformation retract of $\mathcal{X}_n$ to $K_n$. ($K_n$ is called *the spine of* $\mathcal{X}_n$). The spine was used to get the "right" virtual cohomological dimension for $\mathrm{Out}(F_n)$.

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