ON THE GEOMETRY OF OUTER SPACE

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1. Outer Space

Culler and Vogtmann introduced Outer Space in [CV86] as a topological model for the group $\mathrm{Out}(F_n)$. Points of Outer Space are "marked metric graphs."

Definition 1.1 (Graph, positive edges)**.** A *graph* will mean a connected 1-dimensional cell complex. $V(G)$ will denote the vertex set and $E(G)$ the set of unoriented edges. The degree of a vertex $v \in V(G)$ will be denoted $deg_G(v)$, or $deg(v)$ when G is clear.

For each edge, one may choose an orientation. Once the orientation is fixed that oriented edge e will be called *positive* and the edge with the reverse orientation E will be called *negative*. Given an oriented edge e, $i(e)$ will denote its initial vertex and $ter(e)$ its terminal vertex. A *directed* graph is a graph G with a choice of orientation on each edge.

Given a free group F_n of rank $n \geq 2$, we choose once and for all a free basis $A = \{a_1, \ldots, a_n\}$. Let $R_n = \vee_{i=1}^{r} \mathbb{S}^1$ denote the graph with one vertex and r edges, we call this graph an *n-petaled rose*. We choose once and for all an orientation on R_r and identify each positive edge of R_n with an element of the chosen free basis. Thus, a cyclically reduced word in the basis corresponds to an immersed loop in R_n. Moreover, the set of automorphism Φ of F_n are in 1-1 correspondence with homotopy equivalences of R_n that send the vertex to itself modulo homotopy relative to the vertex.

Definition 1.2 (Marked metric F_n-graph)**.** A *marked F_n-graph* is a pair $x = (\Gamma, m)$ where:

- Γ is a graph such that $deg(v) \geq 3$ for each vertex $v \in V(\Gamma)$.
- $m \colon R_n \to \Gamma$ is a homotopy equivalence, called a *marking*.

A *marked metric graph* is a triple (Γ, m, ℓ) so that (Γ, m) is a marked graph and:

- The map $\ell \colon E(\Gamma) \to \mathbb{R}_+$ is an assignment of lengths to the edges. We require (sometimes) that $\sum_{e \in E(\Gamma)} \ell(e) = 1$. The quantity $vol(\Gamma) = \sum_{e \in E(\Gamma)} \ell(e)$ is called the *volume* of G (when we don't require this then we shall say that the graph is *unnormalized*).

Date: September 12, 2017.

Remark 1.3 (Implicit linear structure). For some applications, for example when talking about maps, we need more structure. A linear atlas for a metric graph (Γ, m) is a collection of maps $\{j_\alpha\}_{\alpha \in A}$, with

(1) there exists an edge e of Γ such that $j_\alpha \colon [0,d] \to e$ with $d \leq \ell(e)$ and which restricts to a homeomorphism on $(0,d)$,
(2) the images of the collection of map $\{j_\alpha\}$ cover Γ, and
(3) the transition maps are linear.

As usuall two atlasses are equivalent if their union is an atlas. Given a linear atlas on a metric graph (Γ, ℓ) one can then describe linear edge paths and more generally linear maps between two such graphs. We shall supress this information from now on.

Define an equivalence relation on marked metric graphs by $(\Gamma, m, \ell) \sim (\Gamma', m', \ell')$ when there exists an isometry $\varphi \colon (\Gamma, \ell) \to (\Gamma', \ell')$ so that m' is homotopic to $\varphi \circ m$.

Definition 1.4 (Underlying set of Outer Space)**.** As a set, the (*rank-n*) *Outer Space* $\mathcal{X}_n$ is the set of equivalence classes of marked metric F_n-graphs.

Exercise 1.5. Prove that this definition is equivalent to the following definition using minimal free simplicial F_n-trees.

An F_n-tree is a tree T along with a homomorphism $\rho \colon F_n \to \mathrm{Aut}(T)$. It is minimal if there is no F_n-invariant subtree. It is simplicial if T is simplicial or equivalently, if the translation length elements in F_n are bounded away from zero. It is free if for all $g \in F_n$, $\rho(g)$ has no fixed point.

The F_n trees (T, ρ), (S, τ) are equivalent if there exists an F_n equivariant homothety $h \colon T \to S$.

Prove there exists a bijective correspondence between the set of marked metric graphs and the set of equivalence classes of minimal simplicial free metric F_n-trees.

Definition 1.6. The (open) *simplex* σ in $\mathcal{X}_n$ corresponding to the marked graph (G, m) is

$$\sigma_{(\Gamma,m)} = \{(\Gamma, m, \ell) \in \mathcal{X}_n \mid vol(\Gamma) = 1\}.$$

By enumerating the edges of Γ, we can identify $\sigma_{(\Gamma,m)}$ with the open simplex

$$S_{|E|} = \left\{ \overrightarrow{v} \in \mathbb{R}_+^{|E|} \;\middle|\; \sum_{i=1}^{|E|} v_i = 1 \right\}.$$

Here $E = E(\Gamma)$. We denote this identification by $n \colon \sigma_{(\Gamma,m)} \to S_{|E|}$.

A face of $\sigma_{(\Gamma,m)}$ is the simplex $\sigma_{(\Gamma',m')}$, where (Γ', m') is obtained by collapsing a forest F in Γ. Note that

$$\sigma_{(\Gamma',m')} = \{(\Gamma, m, \ell) \in \mathcal{X}_n \mid \ell \colon E(\Gamma) \to \mathbb{R}_{\geq 0}, vol(\Gamma) = 1, \text{ and } \forall e \in E(F)\ \ell(e) = 0\}.$$

Outer Space has the structure of an ideal simplicial complex built from open simplices (see [Vog02])

Exercise 1.7. Prove that Outer Space is a complex made of open simplicies, i.e. for any two simplicies σ_1, σ_2, we have $\sigma_1 \cap \sigma_2$ is an open simplex. Moreover, show it is locally finite. What are the maximal and minimal dimensions of simplicies?

The simplicial structure endows Outer Space with a simplicial topology. We shall see in subsequent lectures that topology has two other descriptions as the axes topology and the Gromov-Hausdorff topologies which become distinct from the simplicial topology on the compactification of Outer Space.

As mentioned before, Outer Space is a good topological (and metric) model for $\mathrm{Out}(F_n)$.

Definition 1.8 ($\mathrm{Out}(F_n)$ action)**.** If $\Phi \in \mathrm{Aut}(F_n)$ is an automorphism, let $f_\Phi \colon R_n \to R_n$ be a homotopy equivalence corresponding to Φ via the identification of $E(R_n)$ with the chosen free basis A of F_n. We define a *right action of* $\mathrm{Out}(F_n)$ on $\mathcal{X}_n$. An outer automorphism $[\Phi] \in \mathrm{Out}(F_n)$ acts by $[\Gamma, m, \ell] \cdot [\Phi] = [\Gamma, m \circ f_\Phi, \ell]$.

Clearly, each outer automorphism preserves the simplicial structure of Outer Space and therefore, acts by homeomorphisms.

Definition 1.9 (Reduced Outer Space $\mathcal{R}_n$)**.** The *(rank-n) reduced Outer Space* $\mathcal{R}_n$ is the subcomplex of $\mathcal{X}_n$ consisting of precisely those simplices $\sigma_{(\Gamma,m)}$ such that Γ contains no separating edges. This space is connected and an $\mathrm{Out}(F_n)$-equivariant deformation retract of $\mathcal{X}_n$.

Exercise 1.10.
(1) Show that for $n = 2$ the kernel of the $\mathrm{Out}(F_n)$ action is the outer class of the automorphism $x \to x^{-1}, y \to y^{-1}$.
(2) Show that for $n \geq 3$ the $\mathrm{Out}(F_n)$ action is faithful.
(3) Show that the stabilizer of a point in Outer Space is finite.

Using Stallings' theorem [] that any infinite ended group acts by isometries on a simplicial tree one can show,

Theorem 1.11 (Neilsen Realization Theorem for $\mathrm{Out}(F_n)$)**.** *Every finite subgroup of* $\mathrm{Out}(F_n)$ *fixes a point in Outer Space.*

The first theorem proved about Outer Space was

Theorem 1.12. [CV86] *For* $n \geq 2$ *Outer Space is contractible.*

Corollary 1.13. $\mathrm{Out}(F_n)$ *is finitely presented.*

There is a much stronger corollary:

Definition 1.14 (vcd)**.** Let G be a group. The cohomological dimension of G, $cd(G) = m$ if m is the minimal length of a projective $\mathbb{Z}G$ resolution of $\mathbb{Z}$. If G acts free and properly discontinuously on a contractible simplicial complex X then $cd(G) \leq dim(X)$. If $H < G$ then $cd(H) \leq cd(G)$. Moreover, if H is finite then $cd(H) = \infty$. The virtual cohomological dimension of G is the $cd(H)$ for any (every) H that is a finite index subgroup and torsion free.

Exercise 1.15.
(1) Show that $\mathrm{Out}(F_n)$ has torsion.
(2) Show that the kernel of the homomorphism $\mathrm{Out}(F_n) \to GL(n, \mathbb{Z}/3)$ obtained by abelianizing and then sending each entry to $\mathbb{Z}/3$, is a finite index torsion free subgroup of $\mathrm{Out}(F_n)$.

This contractibility proves that

Corollary 1.16. $vcd(\mathrm{Out}(F_n)) \leq 2n - 3$.

Remark 1.17*.* To get this inequality Culler and Vogtmann use the $\mathrm{Out}(F_n)$ action on the *spine* of $\mathcal{X}_n$ see [CV86] or [Bes12] for more details.

For the other inequality one finds a free abelian subgroup $H < \mathrm{Out}(F_n)$ such that $rank(H) = 2n - 3$.

1.1. **The Lipschitz metric.**

Definition 1.18 (Lipschitz distance)**.** Let $x = (G, m, \ell), y = (G', m', \ell')$ be marked metric graphs, a change of marking from x to y is a linear (see Remark 1.3) map $h\colon (G, \ell) \to (G', \ell')$ so that m' is homotopic to $h \circ m$. The Lipschitz constant of h, $Lip(h)$ is the maximal slope of h on any edge of G. Given $x, y \in \mathcal{X}_n$ then

$$d(x, y) = \log \min\{Lip(h) \mid h \text{ is a change of marking from } x \text{ to } y\}$$

The fact that the minimum in the distance is realized follows from Arzella-Ascolli's theorem. A change of marking h that realizes the minimum is called *optimal*.

The Lipshitz distance is an asymmetric metric, i.e.

(1) For all $x, y \in \mathcal{X}_n$ we have $d(x, y) \geq 0$.
(2) If $d(x, y) = 0$ then $x = y$.
(3) For all $x, y, z \in \mathcal{X}_n$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

Exercise 1.19. Prove the facts stated above.

Definition 1.20 (train track structure)**.** A train track structure on a graph G is an equivalence relation on its set of directions, which satisfies that if two directions are equivalent then they are incident at the same vertex. The equivalence classes are called "gates".

Given a linear map $h\colon (G, \ell) \to (G', \ell')$ it induces the following structure: the directions represented by the edges $e, e' \in E(G)$ with the same initial vertex are equivalent if the first edge in $h(e)$ and the first edge in $h(e')$ coincide.

Definition 1.21. Given G with a train track structure a turn $\{e, e'\}$ is called *legal* if it is not contained in a gate. Otherwise it is called *illegal.* A path α in x is *legal* if it does not map over any illegal turn. Otherwise it is illegal.

Definition 1.22 (The tension graph)**.** Given a linear map $h\colon x \to y$ define x_h to be the subgraph of x that consists of all edges that are stretched by $Lip(h)$ under h (the maximally stretched edges). The tension graph x_h can be equipped with a train track structure induced from h.

Definition 1.23. Let $\alpha \in F_n$ or more generally, a conjugacy class in F_n, and let $x = (G, m, \ell) \in \mathcal{X}_n$. We denote by α_x the immersed loop in x freely homotopic to $m(\alpha)$ and by $\ell_x(\alpha)$ the length of α_x in x.

Lemma 1.24. *(The tension graph lemma) If $h\colon x \to y$ is an optimal change of marking such that among optimal maps it has a minimal tension graph, then each vertex in x_h has at least two gates.*

Proof. Suppose x_h has a vertex v with one gate γ. We perterb h to h' where $Lip(h') \le Lip(h)$ and $x_{h'} \subset x_h$. If γ contains only one edge then change h (to h') locally at v so that $h'(v)$ is ε-close to $h(v)$, the image of the single edge e of γ got shorter in y by ε, and the images of $e' \neq e$ incedent at v got longer by ε. We choose ε small enough so that $\frac{\ell_y(h'(e'))}{\ell_x(h(e'))}$ is still smaller than $Lip(h)$. But now $\frac{\ell_y(h'(e))}{\ell_x(h(e))} < Lip(h)$. Then either $Lip(h') < Lip(h)$ contradicting the fact that h was optimal or $x_{h'} \subset x_h - \{e\}$ contradicting that x_h was minimal.
If γ contains more than one edge then the perturbation is similar. Note that all edges in γ start with the same initial path so the previous strategy will work in this case as well. ☐

Definition 1.25. Given two points $x, y \in \mathcal{X}_n$ the stretch of α from x to y is $s(\alpha, x, y) = \frac{\ell_y(\alpha)}{\ell_x(\alpha)}$.

Corollary 1.26 (alternative definition of Lipschitz distance)**.**

$$d(x, y) = \log \max\{s(\alpha, x, y) \mid \alpha \in F_n\}$$

An α realizing this maximum is called a witness*. Moreoevr, there exists a witness α so that $\ell_x(\alpha) \le 2$.*

Proof. Let h be optimal, i.e. $Lip(h) = \log d(x, y)$. Then for any loop α in x, $\ell_y(\alpha) \le Lip(h)\ell_x(\alpha)$. Equality occurs iff $\alpha \subset h_x$ and is legal. Thus,

$$d(x, y) \ge \log \max\{s(\alpha, x, y) \mid \alpha \in F_n\}$$

Given h as in Lemma 1.24, find a loop α that is legal and crosses all edges once except for possibly, one edge which is crossed twice (once in each direction). This α satisfies the claim. ☐

Exercise 1.27. Prove the claim in the proof of the Corollary 1.26.

Definition 1.28 (candidates)**.** A candidate of x is a loop α in x that is embedded, or in the shape of a figure 8 or is in the shape of a barbell. The same proof as in Corollary 1.26 shows that for all $y \in \mathcal{X}_n$ there exists a candidate α so that $s(\alpha, x, y) = e^{d(x,y)}$

Corollary 1.29 (computing distances)**.**

$$d(x, y) = \log \max\{s(\alpha, x, y) \mid \alpha \in F_n \textit{ so that } \alpha_x \textit{ is a candidate in } x\}$$

Since the number of candidates is finite, this computation is finite.

1.2. **Moving around in Outer Space.** There are two natural ways to move around in Outer Space. When one changes the lengths of edges, without changing the homeomorphism type of the underlying graph or the marking, then the result is to change the point in the simplex.

The other way of moving is called folding. Given a point $x_0 = (G, \mu, \ell)$ let e_1, e_2 be oriented edges in G initiating at the same vertex v. The set $\{e_1, e_2\}$ is called a turn. We describe the outcome of folding the turn $\{e_1, e_2\}$ in x_0. Parametrize the points on these edges by (t, i) for $0 \le t \le \ell(e_i)$ and $i = 1, 2$. The graph x_t is obtained from x_0 by the quotienting the equivalence relation:

$$(s, 1) \sim (s, 2) \text{ for } 0 \le s \le t \le \min\{\ell(e_1), \ell(e_2)\}.$$

This operation can be easily generalized to folding a set of turns, each with its own speed, i.e. if the turn $\{e_1, e_2\}$ is folded with speed a then the equivalence relation on x_t is

$$(as, 1) \sim (as, 2), \quad \text{for } 0 \le s \le t \le \frac{\ell(e_i)}{a}.$$

Note that for the maximal t satisfying the conditions above, the point x_t lies in a different simplex than $x_{t'}$ for $t' < t$. It may happen that x_0 lies on a different simplex than $x_{t'}$ for $t' > 0$.

Exercise 1.30. Let x_t for $0 \le t \le \tau$ be a path obtained from x_0 by folding the turns $T_1, \dots, T_k$ at speeds $a_1, \dots a_k$ prove that the path x_t is a Lipschitz geodesic, meaning, for $0 \le r \le s \le u \le t$ we have

$$d(x_r, x_u) = d(x_r, x_t) + d(x_t, x_u)$$

(one can then reparametrize t so that x_t has unit speed in Outer space).

Concatenating fold segments one after the other, if care is taken, produces a Lipschitz geodesic.

Exercise 1.31. Let α be a conjugacy class in F_n and let $\{x_t\}_{t\in[0,L]}$ be a concatenation of fold segments. Denote by α_t the immersed loop representing α in x_t. If α_t contains no folded turn for all t then for $0 \leq r \leq s \leq u \leq t$ we have

$$d(x_r, x_u) = d(x_r, x_t) + d(x_t, x_u)$$

1.3. **Convexity and Quasi-convexity in Outer Space.**

Definition 1.32. A subset $X \subset Y$ is (weakly) convex if for all $x, x' \in X$ there exists a geodesic $[x, x']$ that is contained in X. It is quasi-convex if for all A, B there exists an R so that any (or there exists an) (A, B)-quasi-geodesic α from $x \in X$ to $x' \in X$ is contained in the R-neighborhood of X.

Geometrically, if X is quasi-convex in Y then X is not very much distorted in Y. A horocycle in the hyperbolic plane is not quasi-convex this copy of $\mathbb{R}$ is very distorted in the whole space. While a geodesic is of course, convex.

There are many ways to get from x to y. Given a change of marking map $h\colon\ x \to y$, let us assume that the tension graph is maximal $x_h = x$. If there is more than one illegal turn, we may chose at what relative speeds to fold them. For example, we may choose to fold one illegal turn at a time. This is somehow the simplest type of fold and the most combinatorial. Its advantage and disadvantage is that the produced geodesic is very similar to the Stallings fold decomposition of h. It's metric properties are not ideal. For example one can show that if y, z are in the outgoing ball of radius 1 from x,

$$B_{out}(x, 1) = \{w \in \mathcal{X}_n \mid d(x, w) < 1\}$$

then there exists such a geodesic that lies outside this ball. One may choose to fold all turns at the same speed. This is called a "greedy" geodesic. Length functions are quasi-convex along greedy geodesics, [] this means that these geodesics will not travel far away from the outgoing ball.

Recently, Qing-Rafi [QR17] proved that outgoing balls are convex relative to "stable" geodesics. This means that there is a way to choose the relative speeds so that $\ell_t(\alpha)$ is convex for all time t along the geodesic $[x, y]_{stable}$.

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