

Warwick School  
*Character varieties and the dynamics of the  
mapping class group*

## 1 Character varieties

1. View  $\mathrm{SL}_2(\mathbb{C})$  as the subvariety of  $\mathbb{C}^4$  defined by the equation  $ad - bc = 1$  where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Show that any polynomial function on  $\mathrm{SL}_2(\mathbb{C})$  which is invariant by conjugation is a polynomial in  $\mathrm{tr} M = a + d$ .
2. Show that for  $A, B \in \mathrm{SL}_2(\mathbb{C})$  one has the formula  $\mathrm{tr} AB + \mathrm{tr} AB^{-1} = \mathrm{tr} A \mathrm{tr} B$ . Let  $t_\gamma \in \mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))]$  be the function  $t_\gamma(\rho) = \mathrm{tr} \rho(\gamma)$ . Show that it is invariant by conjugation and satisfies the relation  $t_{\gamma\delta} + t_{\gamma\delta^{-1}} = t_\gamma t_\delta$ .
3. Let  $S$  be a disc with two holes and  $\Gamma = \pi_1(S) = \langle a, b \rangle$ . Show by induction on the number of double points of  $\gamma$  that  $t_\gamma$  is a polynomial in  $t_a, t_b, t_{ab}$ .
4. Prove that for any  $x, y, z \in \mathbb{C}$  there exists a representation  $\rho : F_2 \rightarrow \mathrm{SL}_2(\mathbb{C})$  such that  $x = \mathrm{tr} \rho(a), y = \mathrm{tr} \rho(b)$  and  $z = \mathrm{tr} \rho(ab)$ . Deduce that there is an isomorphism  $X(F_2) = \mathbb{C}^3$ .
5. Find necessary and sufficient conditions on  $x, y, z$  so that the above representation is reducible.
6. Show that the character variety of the torus  $(S^1)^2$  is  $(\mathbb{C}^*)^2 / \sim$  where  $(x, y) \sim (x^{-1}, y^{-1})$ . Realise this variety as an affine variety in  $\mathbb{C}^3$ .
7. Show that the fundamental group of the figure eight knot is  $\langle u, v \mid uv = uw \rangle$  where  $w = v^{-1}uvu^{-1}$ .
8. Show that this presentation is equivalent to  $\langle a, b, t \mid t^{-1}at = ab, t^{-1}bt = bab \rangle$ . Set  $x = \mathrm{tr} \rho(u), y = \mathrm{tr} \rho(v)$ , and compute  $x_1 = \mathrm{tr} a$  and  $x_2 = \mathrm{tr} b$  in terms of  $x, y$ .

9. Show that  $x_1 + x_2 = x_1 x_2$  and deduce that the character variety of the figure eight knot complement is

$$\{(x, y) \in \mathbb{C}^2 \mid (x^2 - y - 2)(2x^2 + y^2 - x^2 y - y - 1) = 0\}$$

10. Find the ideal points of this variety and try to guess the corresponding incompressible surfaces in the complement of the figure eight knot.
11. Describe the character variety of the Heisenberg 3-manifold  $H(\mathbb{R})/H(\mathbb{Z})$  where  $H(A) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in A \right\}$ .
12. For  $p, q, r$  relatively prime integers, set  $\Sigma(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0, |x|^2 + |y|^2 + |z|^2 = 1\}$ . Show that this is a manifold and that the action  $t \cdot (x, y, z) = (e^{it/p}x, e^{it/q}y, e^{it/r}z)$  induces a circle action whose quotient is a sphere.
13. Deduce that  $\Sigma(p, q, r)$  is a Dehn filling of  $S \times S^1$  where  $S$  is a disc with two holes. Show that  $\pi_1(\Sigma(p, q, r)) = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle$ .
14. Study the character variety of  $\Sigma(p, q, r)$ .