Counting loxodromics for group actions

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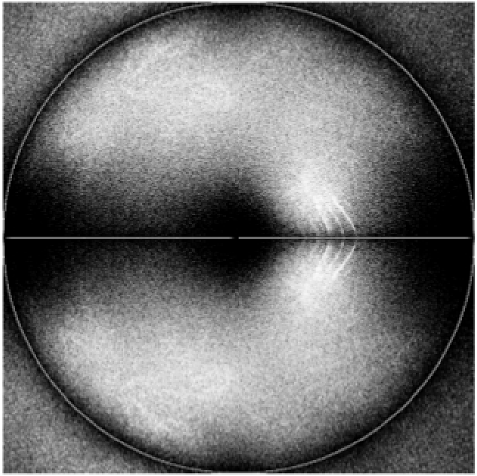

Summary

1. introduction to counting

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joint with Ilya Gekhtman and Sam Taylor

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the counting measure. The set A is generic if

$$P^n(A) \to 1 \quad \text{as } n \to \infty$$

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In general: if you have a boundary for G, for a random walk, you get a hitting measure on ∂G as

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- for $G = Out(F_n)$, $X =$ free factor complex, free splitting complex (Bestvina-Feighn)

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Then

$$\tau(g) > 0 \Leftrightarrow g \text{ is loxodromic}$$

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Recall:
mapping class is pAnosov $\Leftrightarrow$ loxodromic on X = curve complex

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so loxodromics are not generic!

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Question: Can we generalize this result? How far?

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The graph structure is almost semisimple if there exists $c > 0, \lambda > 1$ such that

$$c^{-1}\lambda^n \leq \#S_n \leq c\lambda^n$$

Growth quasitightness

A path γ <u>C-almost contains</u> w if there exists a subpath γ' of γ and two words a, b with $|a|, |b| \leq C$ such that

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Important case: if there is a <u>unique</u> non-trivial component

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Corollary

Suppose that $G < \text{Isom}(\mathbb{H}^n)$ is geometrically finite. Then for any action $G \curvearrowright X$, loxodromic elements are generic.

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Define the right-angled Artin group (RAAG)

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Examples:

- Λ totally disconnected: free groups
- Λ a clique: abelian groups
- "intermediate" cases

Note that $A(\Lambda)$ is a direct product of smaller RAAGs if Λ^{op} is disconnected. If $\Lambda^{op} = \Lambda_1 \sqcup \Lambda_2$, then

$$A(\Lambda) = A(\Lambda_1^{op}) \times A(\Lambda_2^{op})$$

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$$\lim_{n \to \infty} \frac{\#S_n \cap H}{\#S_n} = 0.$$

Scheme of proof for RAAGs

- Modifying Hermiller-Meier, find graph which parameterizes all geodesics in G for the vertex generating set;
- If Λ^{op} is connected, then this graph has a unique recurrent component!
- This immediately implies that the graph structure is growth quasitight
- Moreover, you also get exact exponential growth:

Theorem

There exists $C > 0$, $\lambda > 1$ such that

$$\lim_{n \to \infty} \frac{\# S_n}{\lambda^n} = C$$

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5. to make it into graph structure for Artin group, “double” each vertex

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- By Antolin-Ciobanu, if parabolics P have geodesic graph structure, the whole group G has geodesic graph structure

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Hence, by compactness there is a stationary measure on X^h and one can push it forward to a stationary measure on ∂X. This implies convergence to the boundary á la Furstenberg-Margulis.

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