On Ranks of Hyperbolic Extensions

[jt with Sam Taylor]

Talk at: Geometry of Outer spaces and outer automorphism groups
University Warwick, April 16-20, 2018

10:00, Thurs April 19:

Rank: $rk(G)$ = minimal cardinality of gen set of a group G

- basic, simple minded invariant
- difficult to calculate, even in nice cases!

(Baumslag-Miller-Short '94) rank problem undecidable for hyperbolic groups
$\exists$ families of f-pres grps Q st
- undecidable if Q is trivial
- $rk(Q) \geq 3$ if nontrivial (e.g. $Q = Q_0 * Q_0 * Q_0$)

Contrast: decision problems often easy for hyp grps (e.g. word problem, conj problem, isomorphism problem)

Rips construction '82 $\rightsquigarrow$ $1 \to K_Q \to G_Q \to Q \to 1$ st

- G_Q torsion free, hyp
- $rk(K_Q) \leq 2$

[G_Q = pres for Q + 2 generators and long random relators]

Related to rank is notion of "Nielsen-Equivalence"

Def 2 gen sets $S = \{x_1, \dots, x_n\}$ + $S' = \{x_1', \dots, x_n'\}$ of G are Nielsen-Equivalent if

$$\begin{array}{ccc} F_n & \xrightarrow{\varphi_S} & G \\ \exists \cong \downarrow & \nearrow_{\varphi_{S'}} & \\ F_n & & \end{array}$$

equiv (Nielsen '24)

can transform $\{x_1, \dots, x_n\}$ into S' via moves
- replace $x_i \mapsto x_i^{-1}$
- swap $x_i \leftrightarrow x_j$
- replaces $x_i \mapsto x_i x_j$ some $j \neq i$

Today: Look at (rank + N.E. in) hyperbolic Group Extensions:

SES: $1 \to H \to G \to \Gamma \to 1$ with Γ, H, G inf, hyperbolic

②

Archetypes

• (Thurston) Σ_g surf ~~g~~ genus $g \geq 2$, $f : \Sigma_g \to \Sigma_g$ pseudo-Anosov

mapping torus $M_f \hookleftarrow \Sigma_g$

$$\downarrow$$
$$S^1$$

Thm $1 \to \pi_1(\Sigma_g) \to \pi_1(M_f) \to \mathbb{Z} \to 1$ is hyp

$$\pi_1(M_f) \cong \pi_1(\Sigma_g) \rtimes_{f_*} \mathbb{Z}$$

Thm (Souto '08)

$rk(\pi_1(M_{f^m})) = 2g+1$ for all suff large m.

• (Brinkmann '00, Bestvina-Feighn '92)

If $\phi \in Out(F_n)$ atoroidal (no power ϕ^k fixes any conj class)

⇒ Thm $1 \to F_n \to F_n \rtimes_\phi \mathbb{Z} \to \mathbb{Z} \to 1$ is hyp.

what about ranks

Thm (D-Taylor)

1) Let $f_1, \dots, f_k \in Mod(\Sigma_g)$ be indep pAs. $\exists N$ s.t. for $M_1, \dots, M_k \geq N$,

$\Gamma = \langle f_1^{M_1}, \dots, f_k^{M_k} \rangle$ gives hyp extn

$$1 \to \pi_1(\Sigma_g) \to E_\Gamma \to \Gamma \to 1$$

↳ preimage of Γ in $Mod(\Sigma_g \setminus \{pt\})$

with $rk(E_\Gamma) = 2g + k$ $(= rk(\pi_1(\Sigma_g)) + rk(\Gamma))$

and any min-card gen sets are Nielsen Equiv.

2) same for $\phi_1, \dots, \phi_k \in Out(F_n)$ independent atoroidal iwips

+ E_Γ = preimage of $\Gamma = \langle \phi_1^{M_1}, \dots, \phi_k^{M_k} \rangle$ in $Aut(F_n)$.

Rmk: hyperbolicity already known by Mosher '97 (1) + Bestvina-Feighn-Handel '97

Goal was to generalize Souto, which we did, but discovered that our approach applied in more generality + relied on "Scott Swarup" property.

Def A hyp grp extension $1 \to H \to G \to \Gamma \to 1$ has the Scott Swarup Property (SS) if every f-gen, inf index subgroup of H is quasi-convex in G.

Rmk f-index subgrps of H are never q-convex! ($H \triangleleft G$!)

Thm (Scott-Swarup) If $f: \Sigma_g \to \Sigma_g$ pA, then $1 \to \pi_1(\Sigma_g) \to \pi_1(M_f) \to \mathbb{Z} \to 1$ has (SS).

- very cool, but in fact a general phenomenon?

Thm Suppose $\Gamma \le \mathrm{Mod}(\Sigma_g)$ has orbit map $\Gamma \to \mathcal{C}(\Sigma_g)$ a qi-embedding. Then (Farb-Mosher '02, Hamenstädt '05, Kent-Leininger '08) $1 \to \pi_1(\Sigma_g) \to E_\Gamma \to \Gamma \to 1$ is hyp extn

(D-Kent-Leininger '14) has (SS)

Thm (D-Taylor) Suppose $\Gamma \le \mathrm{Out}(F_n)$ purely atoroidal & orbit map $\Gamma \to \mathcal{F}_n$ a qi-embedding. Then

$1 \to F_n \to E_\Gamma \to \Gamma \to 1$ is hyp, and has (SS)

So: (SS) turns out to be quite common, & is key thing needed to study ranks in general

Main Thm (D-Taylor) Let $1 \to H \to G \to \Gamma \to 1$ be exact seq of inf hyp grps with (SS) & H torsion free.

$\forall r \ \exists N$ s.t. for any subgrp $\Delta \le \Gamma$ with

- $\mathrm{rk}(\Delta) \le r$
- $\|\delta\|_\Gamma \ge N \quad \forall \delta \in \Delta \setminus \{1\}$

$\|\delta\|_\Gamma$: conjugacy length in Γ

the subextension $1 \to H \to G_\Delta \to \Delta \to 1$ has (G_Δ: preimage)

- $\mathrm{rk}(G_\Delta) = \mathrm{rk}(H) + \mathrm{rk}(\Delta)$
- any minimal gen set of G_Δ is N.E. to one in "standard form"

"standard form": contains min gen set of H

- Implies earlier result (can raise to powers)
- Eg: Pass to finite-index $\Delta \le \Gamma$

4

Rmk: SS property is necessary

Ex (Brinkmann) ~~[illegible]~~

atoroidal $\phi \in \mathrm{Aut}(F)$, $F = F_m * \langle a, b \rangle$ st

$\phi(F_m) = F_m$, $\phi(a) = b$, $\phi(b) = w a v$, some $w, v \in F_m$

Thm . $G_\phi = F \rtimes_\phi \mathbb{Z}$ is hyp, but does not have SS

F_m not quasiconvex in $F_m \rtimes_{\phi|_{F_m}} \mathbb{Z}$.
$\wedge$ inf index
F

hence not q-convex in $G\phi$.

· for all k odd, $\phi^k(a) = w_k b v_k$,

$\phi^k(b) = w_k' a v_k'$

$\Rightarrow rk(G_{\phi^k}) \leq m+1 < \mathrm{rank}(F)+1$
gen by $F_m + a$

Hence Thm fails for $1 \to F \to G_\phi \to \mathbb{Z} \to 1$

Sketch of Proof: $1 \to H \to G \to \Gamma \to 1$

G δ-hyp, word length $\|\cdot\|$. Fix $N \gg 0$ + $\Delta \subseteq \Gamma$, st $\forall \delta \in \Delta \setminus \{1\}$, $\gamma \in \Gamma$ $\|\gamma \delta \gamma^{-1}\|_\Gamma \geq N$

use image gen set in Γ

Have to choose carefully, at end.

Lem $H \cong F_n$, $n \geq 3$, or $H \cong \pi_1(\Sigma_g)$, $g \geq 2$

(pf: JSJ decomp of H, if nontrivial, preserved by Γ + find $\gamma \in \Gamma$ fixing inf-index subgrp of H)

$\Rightarrow$ Every subgroup $U \lneq H$ either quasiconvex or has $rk(U) > rk(H)$

by SS

know f index subgrps of F_n + $\pi_1(\Sigma_g)$

Notation: tuple $Y = \{y_1, \dots, y_m\}$ in G.

$\ell(Y) = m$, $\|Y\| = \max_i \|y_i\|$, $c(Y) = \min_{g \in G} \|g Y g^{-1}\|$ "conjugacy length" for the tuple

Lem 2 If tuple $Y \subset G$ has

- $Y \subset G_\Delta$
- $C(Y) \le N$ $\longrightarrow$ forces conjugate of Y into H, normal
- $\ell(Y) < rk(H)$ $\rightarrow$ ensures quasiconvex

Then $U = \langle Y \rangle \le H$ + U q-convex in G.

Lem 3 If $u = \langle Y \rangle$ quasi-convex, $C(Y) \le r$, then

Some conjugate $\hat{u}$ of u is $A(r)$ quasi-convex,

hence $d_{Haus}(\hat{u}, \mathcal{H}(\hat{u})) \le B(r)$

"fat" convex hull of limit set $\Lambda \hat{U} \subset \partial G$

point: finite list of q-convex subgrps that can be generated from ball radius r,

take $A(r)$ = worst q-convexity constant for such groups

Let L = minimal length tuple with $\langle L \rangle = G_\Delta$

know $\ell(L) \le rk(H) + rk(\Delta) \le rk(H) + r$.

Induction Suppose $M = (Y_1, \dots, Y_s ; T)$ partitioned tuple Nielsen equiv. L

st

- $C(Y_i) \le D \ll N$,
- $\langle T \rangle \twoheadrightarrow \Delta$
- $\ell(Y_i) < rk(H)$

base case: $M = (\; ; L)$

Thm (Kapovich-Weidman '05) $\exists K = K(rk(H) + r)$ st.

If $G_\Delta \neq \langle Y_1 \rangle * \cdots * \langle Y_s \rangle * \langle T \rangle$ with $\langle T \rangle$ free, then

M equiv. to $M' = (Y_1', \dots, Y_s' ; T')$ with $C(Y_i') \le C(Y_i)$ +

either:

1) Some $i \neq j$, $d_{Haus}(\mathcal{H}(\langle Y_i' \rangle), \mathcal{H}(\langle Y_j' \rangle)) \le K$
2) Some i and $t \in T'$, $d_{Haus}(\mathcal{H}(\langle Y_i' \rangle), t \cdot \mathcal{H}(\langle Y_i' \rangle)) \le K$
3) Some $t \in T'$ has $C(\{t\}) \le K$.

6

Lem

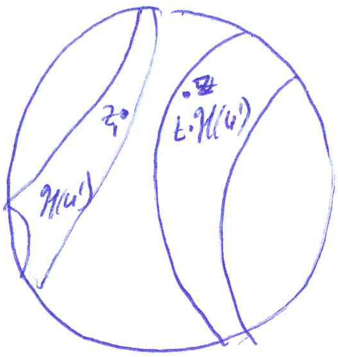
~~pf~~ possibility (1) $\Rightarrow$ (Y_i', Y_j') N.E to Y with $\ell(Y) \leq D' = D'(D)$

possibility (2) $\Rightarrow$ (Y_i', t) N.E to Z with $\ell(Z) \leq D'$

Idea for 2: (1 similar)

$u' = \langle Y_i' \rangle$

$g \in G$ st. $\| g Y_i' g^{-1} \| \leq D$ + $\hat{u} = g \langle Y_i' \rangle g^{-1}$ close to $\mathcal{H}(\hat{u}) = g \cdot \mathcal{H}(\langle Y_i' \rangle)$



$z_1, z_2 \in \mathcal{H}(u')$, $d(z_1, t z_2) \leq K$

$w_1, w_2 \in U'$ st ~~...~~

$$d(g w_i g^{-1}, g z_i) \leq B(D)$$

$$\shortparallel$$

$$d(w_i g^{-1}, z_i)$$

~~..~~ $\Rightarrow$ $d(w_1 g^{-1}, t w_2 g^{-1})$ ~~small~~ $\leq 2B(D) + K$

$$\shortparallel$$

~~...~~ $\| g (w_1^{-1} t w_2) g^{-1} \|$

Thus $Z = (Y_i', w_1^{-1} t w_2)$ NE to (Y_i', t) + $\ell(Z) \leq 2B(D) + K$. ∎

get: M N.E to $\tilde{M} = (\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}}; \tilde{T})$ with ~~...~~ $\ell(\tilde{Y}_i) \leq \tilde{D}(D)$

- $\ell(\tilde{T}) < \ell(T)$ + $\tilde{s} \leq s+1$
- $\ell(\tilde{T}) = \ell(T)$ + $\tilde{s} < s$

Now ~~repeat~~ repeat; after bdd # iterations, must arrive at

$M = (Y; T)$ where $\langle Y \rangle = H$ + $\langle T \rangle$ projects Δ ∎