Conjugacy Problem for Polynomially Growing Elements of $Out(F_n)$
(joint with MICHAEL HANDEL)

① UPG consists of all polynomial growing elements φ of $Out(F_n)$ whose action on $H_1(F_n;\mathbb{Z}_3)$ is trivial.

- φ is polynomially growing if $\forall x \in F_n$, $\exists$ poly P s.t. $\|\varphi^i(x)\| \leq P(i)$.
- $H_1(F_n;\mathbb{Z}_3)$ is finite.

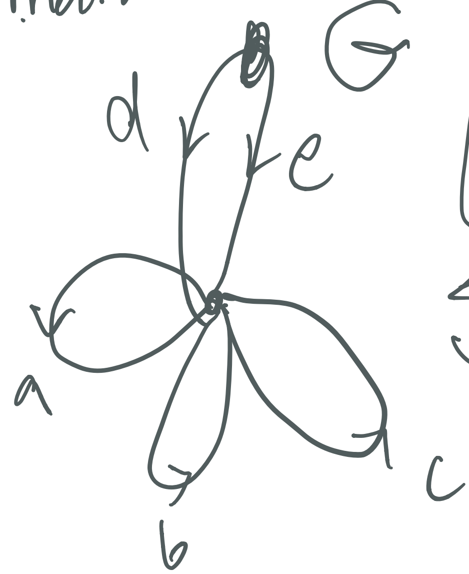
Theorem (F.-Handel) There is an algorithm that given $\varphi, \psi \in UPG$ decides whether $\exists \theta \in Out(F_n)$ s.t. $\varphi^\theta = \psi$ where $\varphi^\theta := \theta\varphi\theta^{-1}$, and if YES produces such a θ.

Cohen - Lustig 1999 solved the conj. problem for linearly growing elements of UPG.

Krstić - Lustig - Vogtmann 2001 solved the conj. problem of linearly growing elements of $Out(F_n)$

(2) Example:

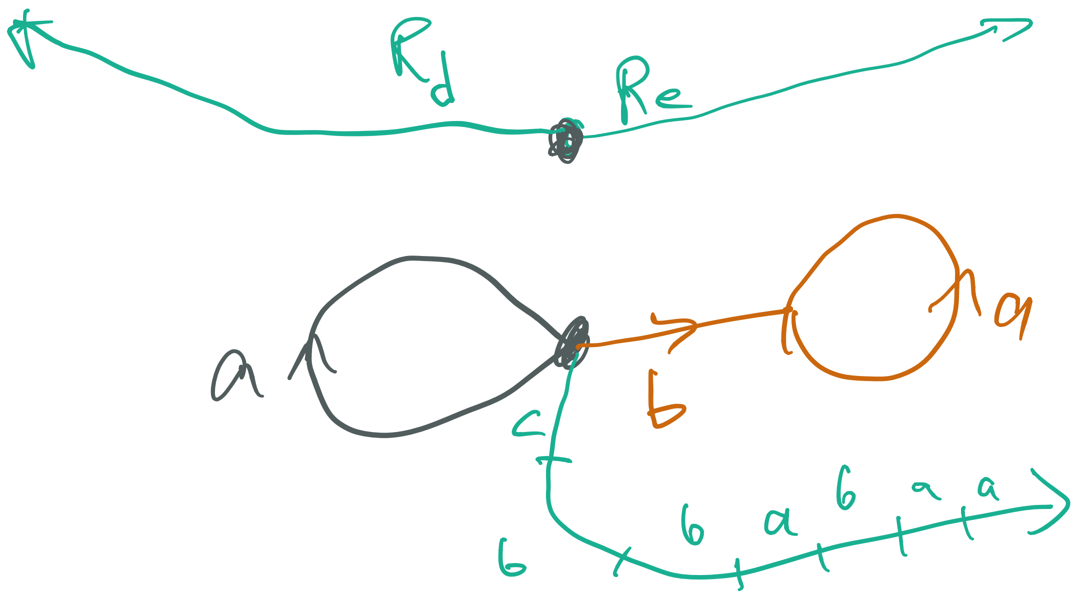
marked graph G



$$a \to a$$
$$b \to ba$$
$$c \to cb$$
$$d \to db^2$$
$$e \to eb^3$$

f

Eigengraph $\Gamma(f)$



- Start with fixed vertices and edges
- For each linear edge, add a lollipop.
- For each higher edge $E \to Eu$, add its eigenray

$$D = E_u f(u) f^2(u) f^3(u) \cdots$$

$R_E = \ldots$

$R_c = cbbaba^2ba^3 \ldots$

<u>Fact</u>: $\Gamma(f)$ carries only ϕ-invariant lines (in fact exactly those with non-repelling endpts), i.e. $\Gamma(\phi)$ is an invariant.

③ This example is general in UPG. Every $\varphi \in UPG$ has a representative $f: G \to G$ where G is constructed as above: start with $\emptyset$ and iteratively add edges $E \to Eu$ where u is a circuit that already exists.

- Can construct the eigengraph $\Gamma(f) = \Gamma(\phi)$ in exactly the same way.

$\Gamma(\phi)$ represents an invariant. Many people starting with Nielsen looked at $\Gamma(\phi)$. (Goldstein-Turner, Gersten, Cooper, ...)

<u>Confession</u> : Some more care should be taken in constructing f. (CT 6).

Theorem (F-H) (Recognition)
$\Gamma(f)$ together with some numerical data is a complete invariant of φ.

- Will ignore numerical data for rest of talk.

Conj. Problem becomes:

$$\Gamma(\phi) \xrightarrow{\exists ?\, \theta} \Gamma(\Psi)$$

meaning θ takes lines in $\Gamma(\phi)$ exactly to the lines in $\Gamma(\Psi)$.

Step 1: Compute

$\Gamma(\phi)$, $\Gamma(\Psi)$

and construct a bijection B preserving types, extend the bijection compatibly to rays.

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(4) Tool (Gersten/Whitehead). $\exists$ algorithm to decide given H_i, H_i' finite sequences of f.g. s/gps of F_n if $\exists\, \theta \in Out(F_n)$ s.t.

$$\theta([H_i]) = [H_i'] \text{ and}$$

if YES produces one θ_0.
Further it produces a finite generating set for the group

$$\{\delta \mid \delta\theta_0([H_i]) = [H_i']\}.$$

Step 2: Ask Gersten/Whitehead if $\exists\, \theta$ s.t. $\theta([\Gamma(\phi)]) = [\Gamma(\psi)]$

where $[\Gamma(\phi)]$ indicates the list of conj. classes of subgroups represented by the components of $\Gamma(\phi)$.

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(5) Want to use G/W to decide if we can take lines to lines.

It is enough to consider a finite set of lines $\mathcal{L}(\varphi)$. Key lines are limit lines of rays.

A limit line L of R_E has the defining property that all finite s/intervals appear infinitely often in R_E.

Example: $c\,b\,b\,a\,b\,a^2\,b\,a^3\,b\,a^4 \ldots$ has 2 limit lines a^∞ and $a^\infty b a^\infty$.

$\Omega(r)$ is the set of limit lines of r.

$\Omega(\phi) = \bigcup_r \Omega(r)$ is the set of limit lines.

Fact: • This collection is finite and consists of ϕ-inv lines.

• The all are carried by $\Gamma(\phi)$

• $L \in \Omega \Rightarrow$ ends of L are either periodic or an end of $\Gamma(\phi)$.

Extend bijection to include limit lines.

$$B : \Omega(\phi) \longrightarrow \Omega(\psi).$$

Want to algebraicize lines.

(6) A canonical sequence of free factor

systems.

- $F_n = F_1 * \cdots * F_k * F$

$\Longrightarrow \{[F_1], [F_2], \dots, [F_k]\} = \mathcal{F}$

is a FFS.

- $\mathcal{F} \sqsubset \mathcal{F}'$ means $\forall\, [F] \in \mathcal{F}$

$\exists\, [F'] \in \mathcal{F}'$ s.t. F is conj. into F'.

Let $f : G \circlearrowleft$ as above.

Build G iteratively starting with fixed and linear edges.

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G.$$

This gives a seq of free factor systems (G_i is a s/gph of G) that is "canonical", i.e. there is a finite collection of such sequences that can occur in this construction.

The height of a line L is the first k s.t. L is carried by $\mathcal{T}_k$.

Similarly for rays r.

Observation For limit lines L with a non-periodic end r the height of r < height of L.

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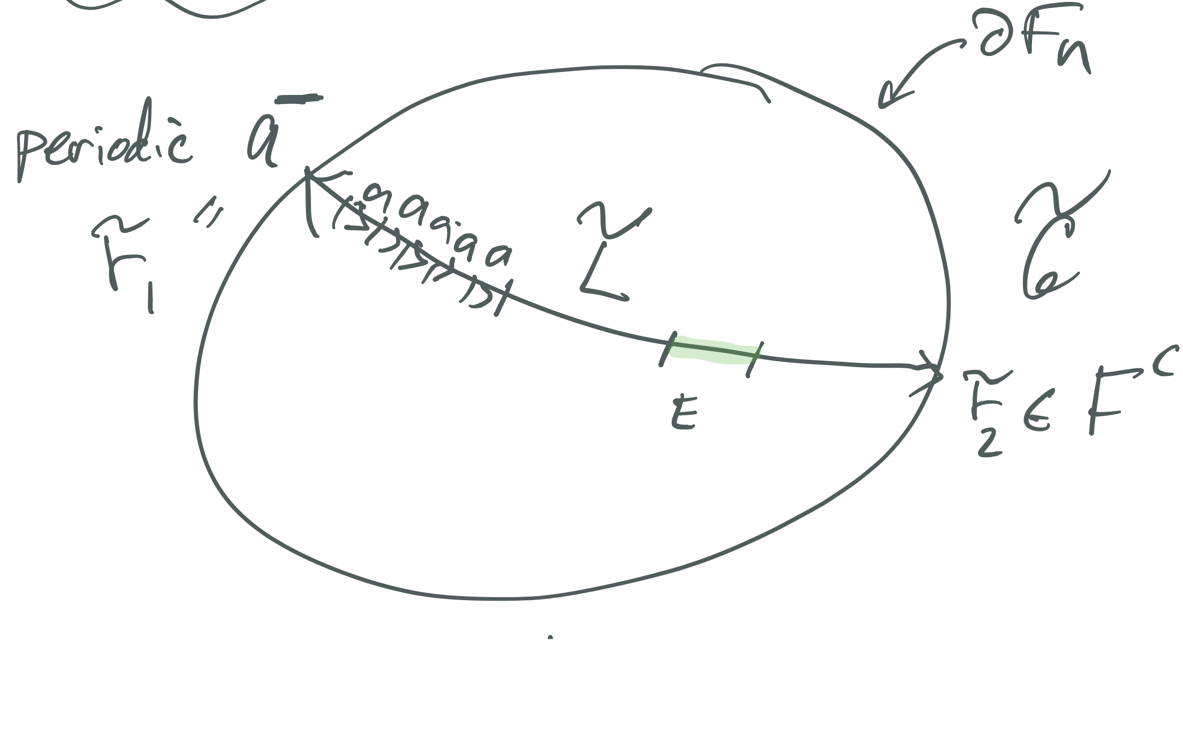
(7) Algebraification of $L \in \Omega$.

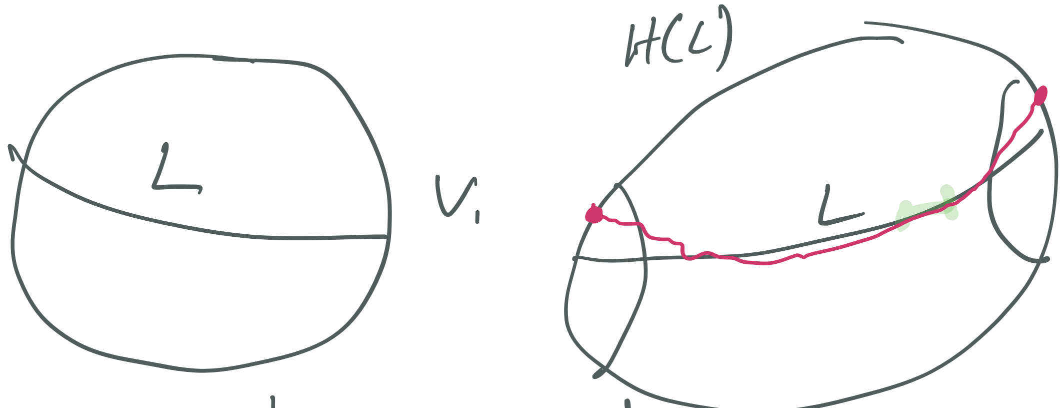


$\tilde{L} = (\tilde{a}, \tilde{r}_2)$

$H(\tilde{L}) := (\langle a \rangle, H(\tilde{r}_2)) = (\langle a \rangle, F^c)$.

where $H(\tilde{r})$ is the lift of smallest height $[F] \in \mathcal{F}_K$ s.t. r is carried by F.

$$H(L) = (H(\tilde{r}_1), H(\tilde{r}_2))^{F_n}$$



Not a strong invariant.

$$\underbrace{L_1, \ldots, L_K}_{\mathcal{L}(\phi)} \xrightarrow{\theta ?} \underbrace{L_1', \ldots, L_K'}_{\mathcal{L}(\psi)}$$

$$H(L_1), \ldots, H(L_K) \xrightarrow{\theta ?} H(L_1'), \ldots, H(L_K')$$

Gersten/
Whitehead
can answer this.

Inductive scheme. $\mathcal{X}(\phi,\psi) = \mathcal{X}(-1)$

$\{\theta \in Out(F_n) \mid \theta(H(L)) = H(B(L))$ for all $L \in \mathcal{L}(\phi).\}$

$\mathcal{X}(k) \subset \mathcal{X}(\phi,\psi)$ consists of θ

s.t. $\theta|\mathcal{F}_k$ conjugates $\phi|\mathcal{F}_k$ to $\psi|\mathcal{F}_k'$

and takes lines in $\mathcal{L}(\phi)$

carried by $\mathcal{F}_k$ to those of ψ

carried by $\mathcal{F}_k'$ respecting B.

Point: $\mathcal{X}(k)$ controls ends of

limit lines L carried by $\mathcal{F}_{k+1}$.

and now $H(L)$ becomes a strong

invariant.

This allows one to carry out the inductive scheme.