Out (RACG) is either large or virtually abelian

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Definition: Γ finite simplicial graph, vertex set V.
The right-angled Coxeter group (RACG) with defining graph Γ has presentation

$$W_\Gamma = \langle V \mid v^2 = 1 \ \forall v \in V; \ uw = wu \iff u,w \text{ adjacent in } \Gamma \rangle$$

Definition: G is large if $\exists H < G$, finite index, and epimorphism $H \twoheadrightarrow F_2$.

Theorem (S-Susse)

Out (W_Γ) is either large or virtually abelian.

Corollary (S-Susse)

Out(W_Γ) has property (T) iff it is finite.

Remark: Note the comparison to the situation for RAAGs.
for example $GL(n,\mathbb{Z})$, $n \geq 3$, is infinite and has property (T).
Also Out (F_5) probably.... [Kaluba - Nowak - Ozawa]

Our result actually applies to a larger class of groups. We can replace W_Γ with any graph product of finite abelian groups.

Definition: Γ finite simplicial graph, vertex set V.
$\{G_v : v \in V\}$ set of groups ("vertex groups")

The graph product of $\{G_v\}$ is the quotient of the free product $\ast_{v \in V} G_v$ obtained by adding relations that when u and v are adjacent in Γ, G_u and G_v commute.

When each G_v is abelian it can be broken up into a clique of groups that are either $\mathbb{Z}$ or $\mathbb{Z}_p$ for a prime power p.

Definition: $o : \Gamma \to \mathbb{N} \cup \{\infty\}$, $o(v)$ either a prime power or ∞.

$$G(\Gamma, o) = \langle V \mid v^{o(v)} = 1 \ \forall v \in V, \ o(v) < \infty; \ uw = wu \iff u,w \text{ adjacent} \rangle$$

<u>Fact</u>: every graph product of abelian groups is isomorphic to $G(\Gamma, o)$ for some (Γ, o).

(Examples: RAAG - $o(v) = \infty \ \forall v$, RACG - $o(v) = 2 \ \forall v$.)

We work with $G = G(\Gamma, o)$ today.

Automorphisms

(Corredor - Gutierrez 2012) $\mathrm{Aut}(G)$ is generated by the following set of automorphisms.

1. Labelled graph symmetries - symmetries of Γ that preserve o.

2. Factor automorphisms - fix all vertex groups except one, which you act on by an automorphism
   - eg. inversion of a vertex in $\mathrm{Out}(A_\Gamma)$

3. Transvections - the right transvection R_u^v sends u to uv and fixes all other vertices of Γ.
   - R_u^v is an automorphism iff either
     - $\mathrm{lk}(u) \subseteq \mathrm{st}(v)$ and $o(u) = \infty$,
     - $\mathrm{st}(u) \subseteq \mathrm{st}(v)$ and $o(v) \mid o(u)$.

4. Partial conjugations - fix a multiplier $v \in V$ and C, a (union of) connected component(s) of $V - \mathrm{st}(v)$.

$$\chi_C^v : \begin{cases} x \longmapsto v x v^{-1} & \text{if } x \in C \\ x \longmapsto x & \text{otherwise} \end{cases}$$

Preorders on Γ

Define $u \leq v$ iff $R_u^v \in \mathrm{Aut}(G)$.

Define $u \leq_\infty v$ iff $\mathrm{st}(u) \subseteq \mathrm{st}(v)$ and $o(v) \mid o(u)$.

$\implies R_u^v \in \mathrm{Aut}(G)$.

Finite-index subgroups

<u>Definition</u>: • $\mathrm{Aut}^0(G)$ is the subgroup generated by all partial conjugations and transvections.

• $\mathrm{Aut}'(G)$ is the subgroup generated by all partial conjugations and transvections R_u^v with $u \leq_\infty v$.

→ In $\mathrm{Aut}'(G)$ we throw out all "finite order-on-finite order" transvections.

Let $\mathrm{Out}^0(G)$ and $\mathrm{Out}'(G)$ denote the images in $\mathrm{Out}(G)$.

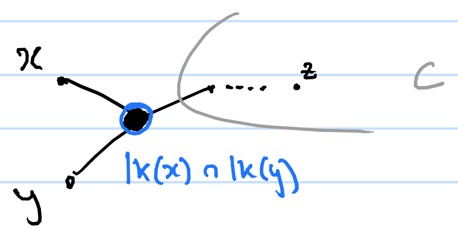
<u>Proposition</u>: (Mühlherr '98 (RACG) ; Gutierrez-Piggott-Ruane '12 ($o(v) < \infty \ \forall v$))

$\mathrm{Aut}^0(G)$ and $\mathrm{Aut}'(G)$ have finite index in $\mathrm{Aut}(G)$

In particular, if $o(v) < \infty \ \forall v \in V$, then the partial conjugations generate a finite index subgroup.

Separating Intersections of Links (SILs)

<u>Definition</u>: A SIL is a triple $(x, y \,/\, z)$ such that $[x,y] \neq 1$ and z is separated from $\{x,y\}$ by $lk(x) \cap lk(y)$:



ie, Z is contained in a connected component C of $\Gamma \setminus lk(x) \cap lk(y)$ which does not contain x or y.

Then we can define partial conjugations χ_C^x and χ_C^y.

FACT: $\langle \chi_C^x, \chi_C^y \rangle \cong \langle x, y \rangle \cong \mathbb{Z}_{o(x)} * \mathbb{Z}_{o(y)}$ $\quad [\mathbb{Z}_\infty = \mathbb{Z}]$.

Some SIL-related facts

(Day '11) $\mathrm{Out}(A_\Gamma)$ contains a subgroup isomorphic to F_2 iff either

(1) Under $\leq$ there is an equivalence class of size ≥ 2
(2) Γ contains a SIL.

Otherwise $\mathrm{Out}(A_\Gamma)$ is virtually nilpotent.

(Gutierrez – Piggott – Ruane '12) Suppose $o(v) < \infty$ $\forall v \in V$.

Then $\mathrm{Out}(G)$ is infinite iff Γ contains a SIL.

Otherwise $\mathrm{Out}(G)$ is finite abelian.

(Charney – Ruane – Stambaugh – Vijayan '10)

If Γ has no SIL, $\mathrm{Aut}^{pc}(G)$ is a graph product of abelian groups.

SILs and quotients

(Guirardel – S '17)

1. If Γ has no SIL then there is a short-exact sequence

$$1 \longrightarrow N \longrightarrow \mathrm{Out}^0(A_\Gamma) \longrightarrow \prod_{i=1}^{k} SL(n_i, \mathbb{Z}) \longrightarrow 1$$

where N is finitely generated nilpotent,
$\{n_i\}$ is the set of sizes of $\leq$-equivalence classes (with more than one element).

2. Suppose each $\leq$-equivalence class in Γ generates an abelian subgroup of A_Γ (note, such subgroups are either free or abelian).

If Γ contains a SIL then $\mathrm{Out}(A_\Gamma)$ is large.

Remark: This is a first step towards a large vs. non-large dichotomy for $\mathrm{Out}(A_\Gamma)$. We do have a "vast vs. non-vast" dichotomy, where large $\not\Rightarrow$ "vast".

Idea for 2.

Find a "special" SIL $(x, y \mid z)$ where we can define

$$Out^{\circ}(A_{\Gamma}) \longrightarrow Out^{\circ}(A_S)$$

where A_S is generated by S, the union of the $\leq$-equivalence classes of x, y and z.

Then verify that the image in $Out^{\circ}(A_S) \cong Out^{\circ}(\mathbb{Z}^a * \mathbb{Z}^b * \mathbb{Z}^c)$ is large.

<u>Special SILs</u> – factor and restriction maps

Let Γ' be an induced subgraph of Γ, and $o|_{\Gamma'} = o'$.
Define $\quad K : G(\Gamma, o) \longrightarrow G(\Gamma', o')$ by killing vertices not in Γ'.

Try to define $\quad$ fact : $Out^{\circ}(G(\Gamma, o)) \longrightarrow Out^{\circ}(G(\Gamma', o'))$

$$\varphi \longmapsto [K(g) \mapsto K(\varphi(g))]$$

$\longrightarrow$ $\operatorname{Ker} K$ should be preserved by φ.

Transvections can be bad:

Suppose $u \notin \Gamma'$, $v \in \Gamma'$. Then $\operatorname{fact}(R_u^v) : \underset{K(u)}{\overset{\shortparallel}{1}} \longmapsto \underset{K(uv)}{\overset{\shortparallel}{v}}$ .... oops!

But partial conjugations are ok:

If $w \notin \Gamma'$ then $K(x_C^v(w)) = 1$.

So we can always define a factor map to $Out^{\circ}(A_{\leq S})$ where

$$\leq S = \{ v \in V \mid v \leq x, y \text{ or } z \}.$$

Definition: A SIL $(x_1, x_2 \mid x_3)$ is special if

(1) The $\leq$-equivalence classes $[x_i]$ are abelian,

(2) We can define

$$\begin{tikzcd} \mathrm{Out}^0(A_\Gamma) \arrow[rr] \arrow[dr, "\mathrm{fact}"'] & & \mathrm{Out}^0(A_S) \\ & \mathrm{Out}^0(A_{\leq S}) \arrow[ur, "\mathrm{res}"'] & \end{tikzcd}$$

ie, Need that the image of fact preserves A_S

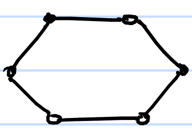
$\Longleftrightarrow \begin{cases} 1. \ x_i \leq u \leq x_j \Rightarrow u \in S = [x_1] \cup [x_2] \cup [x_3] \\ 2. \ u \in {\leq S} \setminus S \Rightarrow \exists \text{ connected component } C \text{ of } \Gamma \setminus st(u) \\ \quad \text{such that } S \subseteq C \cup st(u). \end{cases}$

Lemma (Gutierrez-Piggott-Ruane 2012)

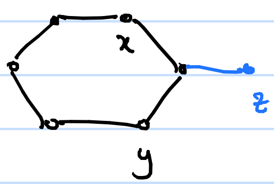
Suppose $o(v) < \infty$ $\forall v \in V$. Then every SIL is special.

Moreover, for every induced subgraph Γ' in Γ, we get a factor map $\mathrm{fact}: \mathrm{Out}'(G(\Gamma, o)) \to \mathrm{Out}'(G(\Gamma', o'))$.

Examples: Assume $o(v) < \infty$ $\forall v$.



no SIL $\Rightarrow$ $\mathrm{Out}(G)$ is finite



$(x, y \mid z)$ is a SIL. Take factor map

$\mathrm{fact}: \mathrm{Out}'(G) \to \mathrm{Out}'(\langle x, y, z \rangle)$

In the image we have χ_z^x, χ_z^y, but nothing else.

So $\mathrm{Im}(\mathrm{fact}) \cong \langle x, y \rangle \cong \mathbb{Z}_{o(x)} * \mathbb{Z}_{o(y)}$.

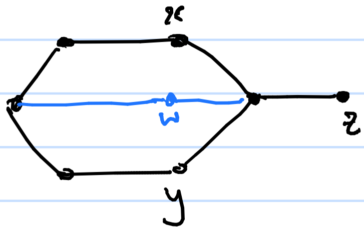
If either $o(x), o(y) > 2 \longrightarrow \mathrm{Out}(G)$ is large.

Definition: A non-Coxeter SIL is a SIL $(x, y \mid z)$ in which either $o(x) > 2$ or $o(y) > 2$.

If (Γ, o) contains a non-Coxeter SIL, then $\mathrm{Out}(G)$ is large.

So we assume $o(x) = o(y) = 2$

Examples:



fact : $\mathrm{Out}'(G) \longrightarrow \mathrm{Out}'(\langle w, x, y, z \rangle)$

Image contains $\chi^{x}_{z}, \chi^{y}_{z}, \chi^{w}_{z}$ and nothing else.

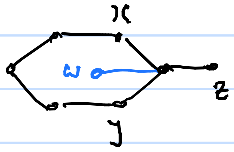
So $\mathrm{Im}(\text{fact}) \cong \langle x, y, w \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 * \mathbb{Z}_2$

Definition: A separating triple intersection of links (STIL) is a quadruple $(w, x, y \mid z)$ such that

- $lk(w) \cap lk(x) \cap lk(y)$ separates z from $\{w, x, y\}$
- between $\{w, x, y\}$ there is at most one edge.

Remark: The second condition ensures that $\langle w, x, y \rangle$ is virtually free.

Example:



$$\left.\begin{array}{l} (w, y \mid z) \\ (y, z \mid w) \\ (z, w \mid y) \end{array}\right\} \text{ all are SILs.}$$

Then fact : $\mathrm{Out}'(G) \longrightarrow \mathrm{Out}'(\langle w, y, z \rangle)$

$\chi^{w}_{z}, \chi^{w}_{y}, \chi^{z}_{w}$ generate the image $\longrightarrow$ fact is surjective.

(Collins '88) If G_1, G_2, G_3 are finite then $\mathrm{vcd}(\mathrm{Out}(G_1 * G_2 * G_3)) = 1$
$\Rightarrow \mathrm{Out}(G_1 * G_2 * G_3)$ is virtually free.

Definition: A *flexible SIL* is a triple $\{x_1, x_2, x_3\}$ such that $(x_i, x_j \mid x_k)$ is a SIL for any $\{i,j,k\} = \{1,2,3\}$.

Theorem: (S-Susse)

Suppose $o(v) < \infty$ $\forall v$. Then $Out(G)$ is large iff either

- Γ contains a non-Coxeter SIL
- Γ contains a STIL
- Γ contains an FSIL.

Otherwise $Out(G)$ is virtually abelian

Remark: For the "otherwise" case, we show that the derived subgroup of $Out(G)$ is abelian, by showing pairs of commutators of partial conjugations commute when there is no non-Coxeter SIL, STIL or FSIL.

Corollary:

If $o(v) < \infty$ $\forall v$, then $Out(G)$ has property (T) iff it is finite.