Automorphisms of hyperbolic groups and growth

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Let G be a f.g gp, S finite gen. set of G

Q1) Let $\varphi \in \mathrm{Aut}(G)$
What are the possible growth types of $|\varphi^n(g)|_S$ avec $n \to +\infty$ and as g varies in G?

Q2) Let $\Phi \in \mathrm{Out}(G)$
What are the possible growth types of $\|\Phi^n(g)\|_S$ as $n \to +\infty$ and as g varies in G?

$$\|\Phi^n(g)\|_S = \inf_{g' \text{ conj. } g} |\varphi^n(g')|_S$$

Ex 0: $G = \mathbb{Z}^N$, $\mathrm{Out}(G) = GL(N,\mathbb{Z}) \rightsquigarrow \|A^n v\| \sim C\lambda^n n^r$ (finitely many possible growth types)

Rk: In general, not much is known

Open: Find a gp G and an automorphism φ of G for which there exists an elt $g \in G$ that grows slower than any exponential but faster than any polynomial under iteration of φ.

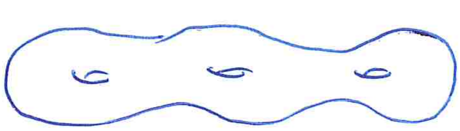
Today, G = torsion-free Gromov hyp gp

Rk ("Q2 $\Rightarrow$ Q1"): Let $\varphi \in \mathrm{Aut}(G)$

$\hat{G} := G * \underset{\langle t \rangle}{\mathbb{Z}}$; $\hat{\varphi} \in \mathrm{Aut}(\hat{G})$ defined by $\begin{cases} \hat{\varphi}(t) = t \\ \hat{\varphi}_{|G} = \varphi \end{cases}$; $\hat{\Phi} \in \mathrm{Out}(G)$

$\hat{S} := S \cup \{t\}$

Then $\forall g \in G$, $\|\hat{\Phi}^n(tg)\|_{\hat{S}} = \|t\varphi^n(g)\|_{\hat{S}} = 1 + |\varphi^n(g)|_S$

Ex 1: $G = \pi_1\Sigma$, Σ =  closed, orientable, $g \geq 2$

$\mathrm{Out}(G) \underset{\text{Dehn-Nielsen-Baer}}{\simeq} \mathrm{Mod}^{\pm}(\Sigma)$

Th (Thurston): $\forall \Phi \in \mathrm{Mod}(\Sigma)$, $\exists \underbrace{\lambda_1, \dots, \lambda_k}_{\text{alg. int.}} \geq 1$ (with $k \leq k(\Sigma)$) st

1) $\forall c \in \pi_1\Sigma$, $\exists i \in \{1, \dots, k\}$ st $\sqrt[n]{\|\Phi^n(c)\|} \longrightarrow \lambda_i$
2) If $\sqrt[n]{\|\Phi^n(c)\|} \longrightarrow 1$, then $\|\Phi^n(c)\|$ grows linearly fast or stays bounded.

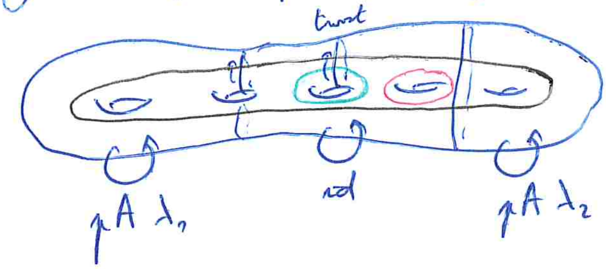
Idea of Thurston's proof

1) If Φ is pseudo-Anosov, then $\forall c \in \pi_1\Sigma \setminus \{1\}$, $\sqrt[n]{\|\Phi^n(c)\|} \longrightarrow \lambda(\Phi) > 1$

$\uparrow$ stretching factor of Φ

[proved by looking at the measured foliations fixed by Φ]

2) In general, up to replacing Φ by Φ^k,



Φ-inv't decomposition

Ex 2: $G = F_N$

Th (Bestvina-Handel, Levitt): $\forall \Phi \in \mathrm{Out}(F_N)$, $\exists\, \lambda_1, \dots, \overset{\text{alg. int.}}{\lambda_k} \geq 1$ $\left(k \leq \frac{3N-2}{4} + 1\right)$ st

1) $\forall g \in F_N$, $\exists\, i \in \{1, \dots, k\}$ st $\sqrt[n]{\|\Phi^n(g)\|} \longrightarrow \lambda_i$

2) If $\sqrt[n]{\|\Phi^n(g)\|} \longrightarrow 1$, then $\|\Phi^n(g)\|$ grows polynomially fast.

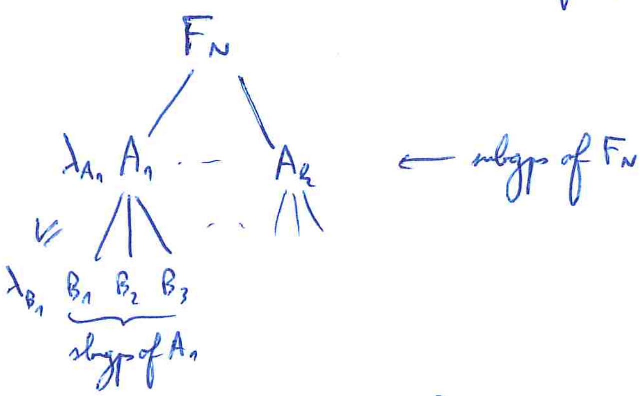
In fact, $\exists C$, $\forall g \in F_N$, $\exists\, p \leq N-1$, $\frac{1}{C} n^p \lambda_i^n \leq \|\Phi^n(g)\| \leq C n^p \lambda_i^n$

(proof relies on train-track theory)

Rk 1: Polynomial growth of higher degree can happen.

Ex in F_3: $\Phi : \begin{cases} c \mapsto cb \\ b \mapsto ba \\ a \mapsto a \end{cases}$ $\longrightarrow$ c grows quadratically

Rk 2: To any $\Phi \in \mathrm{Out}(F_N)$, one can associate a "tree of exponential growth rates":



If g is conjugate into A_1 but not into any of its descendants, then $\sqrt[n]{\|\Phi^n(g)\|} \longrightarrow \lambda_{A_1}$

Ex: F_4

$\Phi : \begin{cases} a \mapsto ab \\ b \mapsto bcab \\ c \mapsto d \\ d \mapsto cd \end{cases}$

$$\begin{array}{cc} F_4 & \lambda_1 \\ \cup\!\!\!| & \vee\!\!\!| \\ \langle c,d \rangle & \lambda_2 \\ | & \vee\!\!\!| \\ \langle [c,d] \rangle & 1 \end{array}$$

Th (Coulon - Hilion - H - Levitt) : Let G be a torsion-free Gromov hyperbolic group.

$\forall \Phi \in \mathrm{Out}(G)$, $\exists \lambda_1, \dots, \lambda_k \geq 1$ alg. int ($k \leq k(G)$) st
$\forall g \in F_N$, $\exists i \in \{1, \dots, k\}$ st $\sqrt[n]{\|\Phi^n(g)\|} \longrightarrow \lambda_i$

(+ tree of growth rates)

Work in progress : If $\sqrt[n]{\|\Phi^n(g)\|} \longrightarrow 1$, then $\|\Phi^n(g)\|$ grows polynomially fast
(And if G is one-ended, then $\|\Phi^n(g)\|$ grows linearly fast or stays bounded)

Sketch of proof of the main theorem (assume Φ has infinite order)

Step 1 : Understand the maximal growth rate

Key prop : $\exists$ nontrivial G-action by isometries on an $\mathbb{R}$-tree T, $\exists \lambda \geq 1$ st

1) $\forall g \in G$ st $\underbrace{\|g\|_T}_{\inf_{x \in T} d(x, gx)} > 0$, $\sqrt[n]{\|\Phi^n(g)\|} \longrightarrow \lambda$

2) $\forall g \in G$, $\limsup \sqrt[n]{\|\Phi^n(g)\|} \leq \lambda$

In addition T is "Φ-invariant" in the weak sense that Φ acts by biLipschitz homeos on T
(in particular the conjugacy classes of point stabilizers in T are Φ-invariant)

Step 2 : Induction : Point stabilizers in T are "simpler" for a good notion of complexity, defined in terms of splittings of G over $\mathbb{Z}$
(if $G = F_N$, one can take the rank for the complexity).

Proof of the key proposition

X = Cayley graph of G wrt S $\qquad G \curvearrowright X \qquad \|g\|_X = \|g\|_S$

$X.\Phi$ = "X with the Φ-twisted action", i.e. $g \underset{X.\Phi}{\cdot} x = \varphi(g)\, x$ $\qquad \|g\|_{X.\Phi} = \|\Phi(g)\|_S$

(where φ is any representative of Φ in $\mathrm{Aut}(G)$: the choice does not affect $X.\Phi$ up to equivariant isometry)

⋮

$X.\Phi^n$ $\qquad \|g\|_{X.\Phi^n} = \|\Phi^n(g)\|_S$

__Th__ (Paulin, Bestvina): Up to passing to a subsequence, $\exists T$ $\mathbb{R}$-tree equipped with an isometric G-action,

$\exists \lambda_n \to 0$ st $\lambda_n X.\Phi^n \xrightarrow[\text{equivariant Gromov-Hausdorff topology}]{} T$

$\rightsquigarrow \forall g \in G$, $\|g\|_{\lambda_n X.\Phi^n} \longrightarrow \|g\|_T$

$\|g\|_{\lambda_n X.\Phi^n} = \lambda_n \|\Phi^n(g)\|_S$

$\rightsquigarrow$ If $\|g\|_T > 0$, $\|\Phi^n(g)\|_S \sim \dfrac{\|g\|_T}{\lambda_n}$

__Pb__: Only true a priori up to a subsequence!

Our proof relies on the study of the dynamics of Φ on the following space:

$$\underbrace{X \quad X.\Phi \quad \dots\dots \quad X.\Phi^n \quad \dots}_{K \text{ compact}}$$

$\leftarrow$ set of projective accumulation trees

"Distance" on K : $d(X, X.\Phi^n) = \log \underbrace{\inf \{ \mathrm{Lip}(f) \mid f : X \longrightarrow X\Phi^n \ G\text{-equiv} \}}_{\mathrm{Lip}(X, X\Phi)}$

Also, if T is a G-space, define a "horofunction" $h_T(X.\Phi^n) = \log \mathrm{Lip}(X.\Phi^n, T)$ $(-\log \mathrm{Lip}(X,T))$

__Drift__: $\log \lambda := \lim_{n \to +\infty} \frac{1}{n} d(X, X.\Phi^n)$ (exists by subadditivity)

__Obs__: The drift gives an upper bound to the growth:

$\forall g \in G$, $\|g\|_{X.\Phi^n} \leq \underbrace{\mathrm{Lip}(X, X.\Phi^n)}_{\sim \lambda^n} \|g\|_X$

$\|g\|_{X.\Phi^n} = \|\Phi^n(g)\|_S$

$\longrightarrow \forall g \in G$, $\limsup \sqrt[n]{\|\Phi^n(g)\|} \leq \lambda$

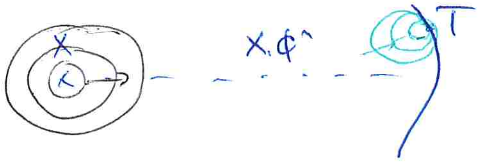
We wish to show that for a random tree T, all elements $g \in G$ acting hyperbolically on T, grow at speed at least λ under iteration of Φ

Define the following proba on K:

ν = weak acc pt of $\frac{1}{n} \sum_{i=1}^{n} \delta_{X.\Phi^i}$ (i.e. $\forall f : K \xrightarrow{C^0} \mathbb{R}$, $\int f \, d\nu = \lim \frac{1}{n_k} \sum_{i=1}^{n_k} f(X.\Phi^i)$)

Lemma (Karlsson – Ledrappier): For ν-a.e. $T \in K$, $\log \lambda = \lim_{n\to\infty} -\frac{1}{n}\, \ell_T(X.\Phi^n)$

["The speed at which the $X.\Phi^n$ is escaping from the basepoint X (the drift $\log\lambda$) is also equal to the speed at which it is approaching T (measured in terms of the horofunction ℓ_T)"]



Pf that lemma $\Rightarrow$ key prop:

$$\forall g \in G, \quad \|g\|_T \leq \underbrace{\mathrm{Lip}(X.\Phi^n, T)}_{\sim \lambda^{-n}} \; \|g\|_{X.\Phi^n}$$

$$\|g\|_{X.\Phi^n} = \|\Phi^n(g)\|$$

$\to$ If $\|g\|_T > 0$, then $\liminf \sqrt[n]{\|\Phi^n(g)\|} \geq \lambda$

Pf of lemma: Let $f : K \xrightarrow{C^0} \mathbb{R}$, $T \longmapsto \ell_T(X) - \ell_T(X.\Phi)$ ("how much have we approached T in one step of the process?")

1)
$$\int f\, d\nu = \lim \frac{1}{n} \sum_{i=1}^{n} f(X.\Phi^i) \qquad (\text{def of } \nu)$$
$$= \lim \frac{1}{n} \sum_{i=1}^{n} \left(d(X, X.\Phi^i) - d(X, X.\Phi^{i-1})\right)$$
$$= \lim \frac{1}{n}\, d(X, X.\Phi^n)$$
$$= \log \lambda$$

2)
$$\int f\, d\nu = \lim \frac{1}{n} \sum_{i=0}^{n-1} f(\Phi^i T) \quad \text{for } \nu\text{-a.e. } T \qquad (\text{Birkhoff})$$
$$= \lim \frac{1}{n} \sum_{i=0}^{n-1} \left(\ell_T(X.\Phi^i) - \ell_T(X.\Phi^{i+1})\right)$$
$$= \lim -\frac{1}{n}\, \ell_T(X.\Phi^n)$$

[Rk: If ν fails to be ergodic, decompose into ergodic components]