Gilbert Levitt - On automorphisms leaving a random subgroup invariant

(joint w/ V. Guirardel)

General question: given $H < G$, -understand automorphisms of G preserving H, denoted $\mathrm{Aut}(G,H)$

-understand automorphisms of H extending to G

Specific question: $G = F_N$, H f.g. is $\mathrm{Aut}(G,H)$ f.g.?

Yes, if H malnormal.

Extreme cases:

Thm (G-L) $G = F_N$, $H \neq \mathbb{Z}$, if every autom. of H extends to G, then H is a free factor.

Thm (Miller-Schupp, '87) Given H, $\exists G \supset H$ s.t. only inner automorphisms of H extend to G.

The proof is by small cancellation.

Principle ("slogan"): Fix G. If $\overset{\text{f.g.}}{H} < G$ is sufficiently generic (or random), then very few auto's of G leave H invariant.

False in $\mathbb{Z}^n$? Perhaps the right word is "complicated"

Thm (G-L) $G = F_N$. Fix $p \geq 1$. Pick $g_1, \ldots, g_p$ randomly independently in the ball of radius n. $H = \langle g_1, \ldots, g_p \rangle$ ← i.e. generically

With probability $\to 1$ as $n \to \infty$, all auto's of G preserving H are conjugations by elements of H.

Main ingredient: peak reduction (Whitehead): Whitehead auto's generate $Aut(F_N)$ very nicely.

Lemma (Kapovich-Schupp-Shpilrain, '06)
A random cyclically reduced word g has minimal length in its $Aut(F_N)$-orbit [hence, g has a small stabilizer...].

Idea of proof: suffices to show certain Whitehead auto's increase length of g.

example where length decreases: F_2, $\alpha: \begin{matrix} a \to a \\ b \to ba \end{matrix}$

$$\underbrace{ba}_{+1}\underbrace{ba^{-1}}_{-1}\underbrace{ba^{-1}}_{-1}\underbrace{b}_{+1}\underbrace{ba^{-1}}_{-1} \xrightarrow{\alpha} baabbbab$$

$$9 \qquad\qquad 8$$

g random $\Rightarrow$ frequencies of ba, bb, ba^{-1} are ~ same

Thm (S-L) G hyperbolic relative to slender subgroups $p \geq 1$, $g_1^n, \ldots, g_p^n$ given by random walks on G. $H = \langle g_1^n, \ldots, g_p^n \rangle$. With probability $\to 1$ as $n \to \infty$, {conjugation by elements of H} has finite index in $Aut(G, H)$

2-ingredients:

- Black box: (Maher-Sisto) $G \curvearrowright T$ (tree) acylindrically generically H is free, quasiconvex, malnormal, acts freely on T, ...

- Relation between automorphisms and splittings. Paulin-type theorem: if the index is infinite, then G splits relative to g over a slender group. $\boxed{p=1,\ H=\langle g\rangle}$

Second Principle: Fix G. If g is generic, it is universally hyperbolic. There is no splitting of G relative to g.

False in F_N: $F_N \to \mathbb{Z}$, $g \to 0$. g is elliptic in a splitting over some F_s.

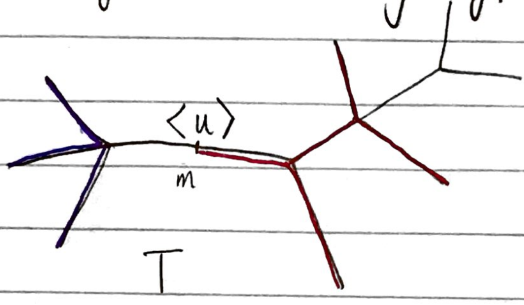
Thm (S-L) G hyperbolic relative to slender groups. g generic $\Rightarrow$ G has no splitting over a slender subgroup relative to g.

Thm $G = F_N$. $g \in G$ cyclically reduced, contains all words of length L. Then F_N does not split relative to g over a subgroup F_s if $s \leq (N-1)(L-2)$

Example: $L=2$ g contains all words of length 2 $\Rightarrow g \not\subset$ proper free factor (Whitehead)

Ex: $L = 3$, Cashen-Manning: $g \in F_N$, g contains all 3-words, F_N has a cyclic splitting rel g.

Proof of Cashen-Manning: $G \curvearrowright T$ with cyclic edge stabilizers. Show g hyperbolic in T.



Fix equivariant $f : \mathrm{Cay} \to T$

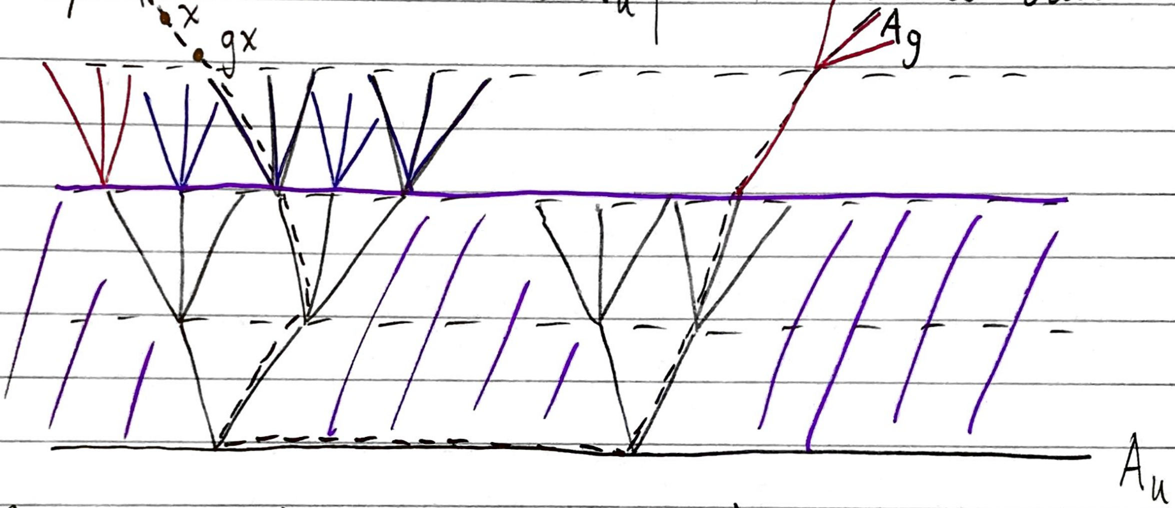
vertex $\mapsto$ vertex

$f^{-1}(m)$ remains at distance $\le R$ from A_u

$R = 2$, below purple line.

Components of $\mathrm{Cay} \setminus$ purple strip are colored red / blue

$\mathrm{Cay}(F_N)$ rooted at axis A_u

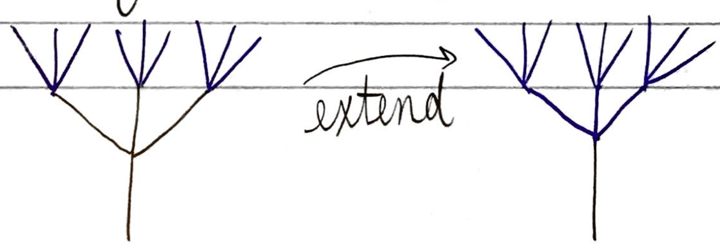


Lemma: If A_g (axis in Cay) has ends of different colors blue/red, then g hyperbolic in T.

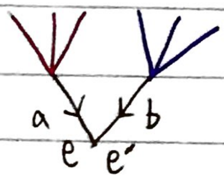
Proof: If not, g fixes a point. $y = f(x)$, $gy = f(gx)$ are blue. g fixes edges containing m. $\forall n \in \mathbb{Z}$, $g^n y$, $g^n x$ are blue, so A_g has no red end. ✓

Try to push coloring all the way to A_u.

If I see this



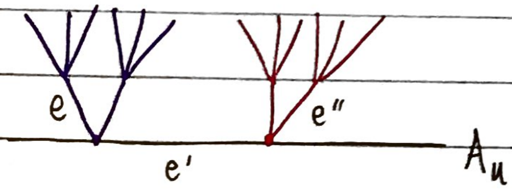
failure:



ab^{-1} appears in g, so some conjugate g' of g has axis $\supset ee'$

lemma $\Rightarrow$ g', and hence g, is hyp. in T.

success:



$ee'e''$ represents a word of length 3.