# On automorphisms groups of RAAGS

joint work with J. Aramayona, J.L. Fernández, P. Fernandez, L. Mendoça

$\Gamma$ a graph. The associated Right Angled Artin Group (RAAG) is

$$A_\Gamma = \langle \underset{\text{vertex}}{v \in \Gamma} \mid [v,w] \ \forall \ w \text{—} v \text{ edge of } \Gamma \rangle$$

The group $\operatorname{Aut} A_\Gamma$ has a well known generating family: the Laurence-Servatius generators:

- Partial conjugations $C_{v,Y}$ $\begin{cases} u \mapsto v^{-1}uv & u \in Y \\ u \mapsto u & \text{otherwise} \end{cases}$ $Y \subset \Gamma - \text{st}$ connected component
- Transvections $\begin{cases} u \mapsto vw \\ u \mapsto u \end{cases}$ $\operatorname{lk} v \subset \operatorname{st} w$
- Inversions $v \mapsto v^{-1}$
- Graphic automorphisms $(\operatorname{Aut} \Gamma)$

We say that an inversion $v \mapsto v^{-1}$ is <u>thin</u> if there is no other $w$ with $v \leq w$, $w \leq v$ (i.e. $[v] = \{v\}$)

($v \leq w$ means $\operatorname{lk} v \leq \operatorname{st} w$).

In this talk, we will consider two difficult open problems about the group $\operatorname{Aut} A_\Gamma$.

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<u>Question 1</u>: When is $\operatorname{Aut} A_\Gamma$ virtually indicable

Recall: a group $G$ is virtually indicable if $\exists \underbrace{H < G}_{\text{fin.}}$ s.t. $\exists H \twoheadrightarrow \mathbb{Z}$

More, for those graphs $\Gamma$ s.t. $\mathrm{Aut}\,A_\Gamma$ is virtually indicable, how big can $b_1(H) = \mathrm{rk}\, H_1(H)$ be?
(for $H \underset{\text{fin. index}}{<} \mathrm{Aut}\,A_\Gamma$ )

- Kazhdan $\Gamma$ complete $\mathrm{Aut}\,A_\Gamma = GL_n\mathbb{Z}$ is <u>not</u> virtually indicable (it has (T))
- Grunewald-Lubotzky $\Gamma = \cdot\ \cdot\ \cdot$ $\mathrm{Aut}\,A_\Gamma = \mathrm{Aut}\,F_3$ is large $\Rightarrow$ virtually indicable
- Open for $\mathrm{Aut}\,F_n$, $n \geq 4$
- Aramayona, MP: gave conditions on $\Gamma$ that imply $b_1(\mathrm{Aut}^* A_\Gamma) > 0$

$\mathrm{Aut}^* A_\Gamma$ = subgroup generated by { partial conjugations, transvections, thin inversions

Main tool: use McCool / Day presentation for $\mathrm{Aut}\,A_\Gamma$ in terms of Whitehead automorphisms (a class of elements which include Laurence-Servatius).

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Now, our objective will be to give some probabilistic estimations of when $b_1(\mathrm{Aut}^* A_\Gamma) > 0$ and how big it can be. But for technical reasons, instead of considering general graphs $\Gamma$ we will restrict ourselves to the case when $\Gamma$ is a tree.

Of course, the real reason to do that is that the class of trees is much easier to handle than the class of graphs, but RAAGs based on trees are an interesting class!

$T$ tree $\Rightarrow A_T = \pi_1(M)$ for $M$ a graph manifold

So, at this point we need an estimation of $b_1(\mathrm{Aut}^0 A_T)$ but also we need something that we can use in probabilistic argument (so it shouldn't be too complicated).

**Theorem A**

$$b_1(\mathrm{Aut}^0 A_T) \geq \sum_{v \text{ deep}} \; \sum_{w \in \mathrm{lk}\, v} (\deg w - 1) = \Upsilon(T)$$

$v \in T$ is deep if $d(v, \partial T) \geq 3$

$\partial T$ = boundary of $T$ = set of leaves

$d$ = edge distance

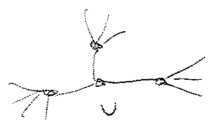
Ex. a deep vertex!

proof: Use Day's presentation to prove that

$\{c_{v,\Omega} \mid v \text{ deep},\ \Omega \subset T - \mathrm{st}\, v \text{ connected component}\}$

is linearly independent in $G/G'$ $\quad G = \mathrm{Aut}^0 A_T$

then count connected components



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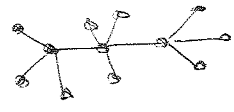
The bound is not sharp at all! It is not difficult to improve it but that would create problems later on

In fact, we can use Day's presentation to give conditions that imply that $b_1(\mathrm{Aut}^{\circ} A_T) = 0$

<u>Lemma</u> Assume that every vertex of $\Gamma$ is

$\begin{cases} \text{either a leaf or} \\ \text{is linked to at least 3 leaves} \end{cases}$

Then $b_1(\mathrm{Aut}^{\circ} A_T) = 0$

<u>Ex</u>: "Hairy tree" 

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Before we state our estimations, we need a technical remark. Instead of working in the class of trees we need to ~~work~~ work in the class of labelled trees. This means that ①—②—③ $\neq$ ①—③—②

$\mathcal{U}_n$ = labelled trees with $n$-nodes

Cayley's theorem: $|\mathcal{U}_n| = n^{n-2}$

We work with a uniform probability distribution

<u>Theorem B</u>

$$\frac{|\text{Trees in } \mathcal{U}_n \text{ with } b_1(\mathrm{Aut}^{\circ} A_T) = 0|}{|\mathcal{U}_n|} \xrightarrow[n \to \infty]{} 0$$

exponentially fast

<u>Theorem C</u> (Expectation)

$$\lim_{n\to\infty} \frac{1}{|\mathcal{U}_n|} \sum_{T \in \mathcal{U}_n} \frac{|\text{Deep vertices}|}{n} = c_3 \simeq 0.35$$

$$\lim_{n\to\infty} \frac{1}{|\mathcal{U}_n|} \sum_{T \in \mathcal{U}_n} \frac{|\Gamma(T)|}{n} = d_3 \simeq 2.07$$

$$\begin{cases} c_3 = \frac{1}{e} e^{-\frac{1}{e}} e^{(e^{1-\frac{1}{e}}-1)/e} \\ d_3 = 2 - \frac{1}{e} + \frac{1}{e}\left(1-\frac{1}{e}\right) e^{1-\frac{1}{e}} \end{cases}$$

the proof of both theorems is based in the so called symbolic method. The idea is to translate questions about combinatorial classes of trees to questions about power series. A key tool is the exponential generating function (egf) of a graded class $\mathcal{A}$:

let $a_n = |\mathcal{A}_n|$, assume it is finite $(n \geq 0)$

the egf of $\mathcal{A}$ is $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$

Manipulation of the class $\longleftrightarrow$ manipulation of the egf

One more technical remark: we work first in the class of rooted labelled trees & make our estimations for the root. Later we use that to deduce our result for labelled trees.

(Theorems A, B, C above are joint work with J. Aramayona, J.L. Fernández and P. Fernández)

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Before we introduce the open question that we will consider in the last part of the talk, we need to define a new subgroup of $\mathrm{Aut}\, A_T$.

the group of symmetric automorphisms of a RAAG $A_\Gamma$ is the group of automorphisms that map every vertex to a conjugate of some other vertex.

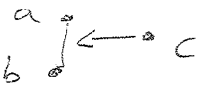
The group of pure symmetric automorphisms is the group of those automorphisms that map each vertex to a conjugate of itself. We denote it by $\mathrm{PAut}\, A_\Gamma$ (sometimes $\mathrm{PSAut} A_\Gamma$)

→ It is a ROAR (see Matt's talk)

→ McCool / Toinet / Koban Piggott
show that it is generated by partial conjugations & find a explicit presentations with

2 types of relators: $[a,b]=1$

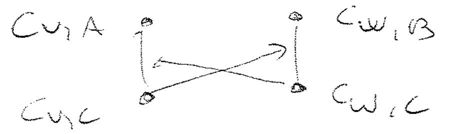
$[c, ab]=1$ (need $[a,b]=1$)

Arrowed graph 


How close is this from being a RAAG?

Recall: $v,w \in \Gamma$ form a SIL pair if

$\exists$ C connected component of both $\Gamma - st\,v$, $\Gamma - st\,w$




| $F_2 \times F_2$

Charney, Ruane, Stambaugh, Vijayan

$\Gamma$ has no SIL pair $\Longrightarrow$ $\mathrm{PAut}\, A_\Gamma$ is a RAAG

$\Longleftarrow$ Ruane & Piggot using BNS invariants

Question 2 : Is $PAutA_\Gamma$ linear? | this part of the talk is joint work with J. Aramayona

Recall : RAAGs are linear , $Aut F_2$ is linear

But $Aut F_n$ is not linear for $n \geq 3$ (Torelli is neither linear)

Many $Aut A_\Gamma$'s are not linear

the obstruction for linearity in these groups is

the Poison subgroup :

$$\sigma = (F_2 \times F_2) *_t \qquad (g,g)^t = (1,g)$$

this group is not linear & is inside $Aut F_n$. But

the standard copy of $\sigma$ in $Aut F_n$ is not in $PAut F_n$

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We construct a representation

$$PAutA_\Gamma \to F_2 \times \cdots \times F_2 \times A_{\tilde{\Gamma}}$$ with subdirect image

But: this can't be faithful!

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Lemma If $\Gamma$ has a SIL pair, then $PAutA_\Gamma$ has

a subgroup isomorphic to

$$\sigma = (F_2 \times F_2) *_t \qquad (g,g)^t = (g,g)$$

Question: Is this group linear? (it is easy to construct representations, the problem is to determine whether they are faithful)

Remark: When we posted our paper on the arxiv, Metaxtsis & Prassidis claimed that they could prove that the answer is yes. However, they haven't posted their preprint so far.

Changing slightly the point of view, we can ask whether we can embed "big" linear groups in $PAut A_\Gamma$. In fact:

Theorem (Aramayona, MP) Let $\Gamma$ be a connected graph. There is another graph $\hat{\Gamma}$ ~~and~~ s.t.

$$A_{\hat{\Gamma}} \trianglelefteq PAut A_\Gamma$$

Moreover, if $\Gamma$ has no SIL pair, then $A_{\hat{\Gamma}} = PAut A_\Gamma$

(in this case $\hat{\Gamma}$ is the graph already constructed by Charney, Ruane, Stambaugh, Vijayan.

Remarks
- $Inn A_\Gamma \leq A_{\hat{\Gamma}}$ and $A_{\hat{\Gamma}} / Inn A_\Gamma$ is free abelian
- Charney, Crisp, Vogtman construct also $K \trianglelefteq PAut A_\Gamma$ free abelian, we have $K \leq A_{\hat{\Gamma}}$
- Question: Is it possible to find bigger RAAGs embeded in $PAut A_\Gamma$?
- $A_{\hat{\Gamma}}$ is also a RoAR (as Matt Day and Ric Wade made clear to me)
- Useful to get lower bounds in vcd

I will just explain how ~~to~~ $A_\Gamma^?$ is defined, the proof of the theorem then follows the lines of the proof by Charney, Ruane, Stambaugh and Vijayan in the no SIL-case.

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Recall that partial conjugations are given as follows:

For each vertex $v$ and each connected component $\Omega$ of $\Gamma - st\,v$, $C_{v,\Omega}$ conjugates vertices of $\Omega$ by $v$ & fixes the rest

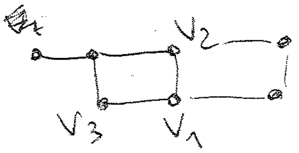
Now, for each $v$ and each $w \in st\,v$

- if the removal of $w$ creates SIL type relations for $v$ we don't remove it
- in other case we remove it

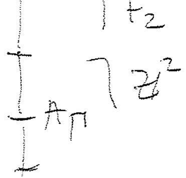
For each connected component after this process, consider $C_{v,\Lambda}$

$A_{\hat\Gamma}$ is the group generated by those $C_{v,\Lambda}$

Example

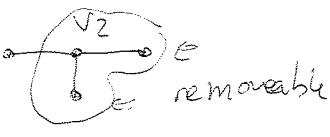



$\Gamma$




We do not have to worry about vertices $v$ with $\Gamma - st\,v$ connected

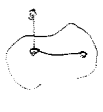
$st\,v_2$




2 connected components

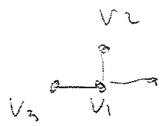

(2 are glued)

$st\,v_3$




only 1 connected component (2 are glued)

$st\,v_1$




2 connected components

All can be removed