$Out(F_n)$ and the handlebody group ('Warwick, April '18)

$Out(F_n)$ is often compared with surface mapping class groups, and studied in comparison to it: similarities and differences from algebra, but mainly geometry

We'll consider a different mapping class group, that of a handlebody

$\#_g S^1 \times D^2 = V_g$, $\mathcal{H}_g = Mcg(V_g)$

We then have: <u>$\mathcal{H}_g$ directly related to Mcg and Out</u>

$$\begin{tikzcd} Mcg(\Sigma_g) & \text{vs.} & Out(F_g) \\ & \mathcal{H}_g \arrow[ul, "r"] \arrow[ur, "a"'] & \end{tikzcd}$$

Example elements:

- meridian Dehn twists (in ker(a))
- handle slides / annulus twists

($\rightsquigarrow$ maps to N.T.)

restriction map r (injective) – V_g aspherical

action-on-π_1-map a (surjective) – realise Nielsen moves

Why the handlebody group? Topology (usually), but for this audience: is $\mathcal{H}_g$ geometrically related to Out or Mcg?

<u>$\mathcal{H}_g$ ~~badly~~ not really related to Mcg and Out</u>

Thm 1 (Hamenstädt-H) For any $g \geq 2$, the map r exponentially distorts distances, i.e. $\forall n\ \exists f_n \in \mathcal{H}_g$ s.t. $\|f_n\|_{\mathcal{H}_g} \geq 2^n$, $\|r(f_n)\|_{Mcg(\Sigma_g)} \leq n$

Thm 2 (Luft, McCullough) $\ker(a)$ is not finitely generated, and is generated by meridian twists.

So, a not immediately accessible via GGT methods

Thm 3 (H?) ~~fact~~ If $n \geq 4$, for no f.i. subgp. $\Gamma < Out(F_g)$ is there a section $s : \Gamma \to \mathcal{H}_g$ to a.

A proof sketch: we will look at ~~left~~ left/right multiplication maps

$$L_w, R_w : \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \longmapsto \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \\ wx_n \text{ or } x_n w \end{matrix} \quad \text{for } w \in \langle x_1, \dots, x_{n-1} \rangle.$$

Then $[L_w, R_{w'}] = 1$ for <u>any</u> choice of w, w'!

Suppose we had a section s, then $\lambda_w = s(L_w)$, $\rho_{w'} = s(R_{w'})$ would commute.

How can elements in $Mcg(\Sigma_g)$ commute?

- [Two pA's commute if (virtually) have common root.] Centralizer of pA virtually cyclic
- reducible elements have (canonical) reduction systems

  $\Sigma = Y_1 \amalg \dots \amalg Y_k$

  st. (up to power) all Y_i preserved, on each pA or id.

  if two commute, need to have the same pA regions and commuting pA's there.

Now: λ_w, ρ_w cannot be pA. $\Rightarrow$ have reduction system α_w, β_w

Since $n \geq 4$, can take $\Theta: \langle x_1, x_2, x_3 \rangle \circlearrowleft$ fully irreducible, nongeometric

Extend to $\Theta: F_4 \circlearrowleft$ by fixing x_4.

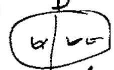
$\varphi = s(\Theta)$. What can φ fix? A loop $\delta \subseteq \Sigma$ fixed gives splitting fixed by Θ ($\langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$, or conj. class $\langle x_n^k \rangle$)

Observe $\varphi^n \lambda_w \varphi^{-n} = \lambda_{\Theta^n(w)}$, has reduction system $\varphi^n \alpha_w$

~~WLOG φ fixes reduction system of λ_w,~~

In order for $[\varphi^n \lambda_w \varphi^{-n}, \rho_w] = 1$ to hold, φ^n cannot move α to intersect β.

$\uparrow$ assume for simpl D



id pA.

but λ_w, ρ_w cannot preserve splitting D (move x_1 into other factor) ∎

Upshot: we cannot (a priori) expect that $\mathcal{H}_g$ has anything ~~geom.~~ in common w/ Out or Mcg.

$\mathcal{H}_g$ nevertheless seems to be related to Out

Good touchstone question: Dehn function. (define!)

Thm (Mosher) $Mcg(\Sigma_g)$ (automatic) hence has quadratic DF

Thm (Bridson–Vogtmann, Handel–Mosher)
$Out(F_n)$, $n \geq 3$, has exponential Dehn function.

Thm (HH)
If $g \geq 3$, then $\mathcal{H}_g$ has exp. DF; if $g = 2$ quadratic.

The genus 3 case

Why is DF of $Out(F_3)$ exponential? Need three automorphisms

$$A: \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \longmapsto \begin{matrix} x_1 \\ x_2 \\ x_1 x_3 \end{matrix}, \quad B: \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \longmapsto \begin{matrix} x_1 \\ x_2 \\ x_3 x_2 \end{matrix}, \quad T: \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \longmapsto \begin{matrix} x_1^2 x_2 \\ x_1 x_2 \\ x_3 \end{matrix}$$

Observe: $[A, T^n B T^{-n}] = 1$, hence

$A T^n B T^{-n} A^{-1} T^n B^{-1} T^{-n} = w_n$ are words of length $4n+4$.
describing trivial element in Aut.

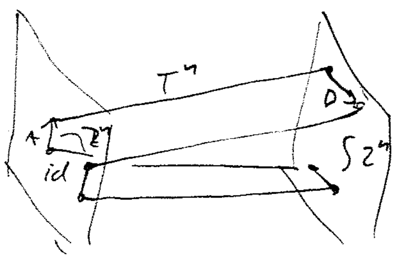
<u>Thm</u> (Bridson–Vogtmann)
The loops defined by w_n are exponentially hard to fill

The proof relies on a similar ~~cycle~~ sequence of words, certifying the exponential DF of $GL(3,\mathbb{Z})$ [Gersten, Epstein–Thurston]

Rough idea: act on homology as $\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$

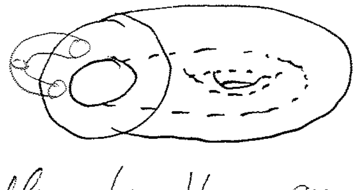
These generate $1 \to \mathbb{Z}^2 \to E \to \mathbb{Z} \to 1$ with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Cycle looks like



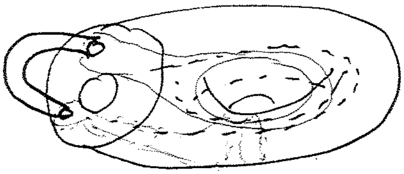
We want to lift the w_n into $\mathcal{H}_3$ so that length stays small; that is enough, as hom. $\mathcal{H}_3 \to Out(F_3)$ decreases area.

- Key trick: $\begin{matrix} x_1 \\ x_2 \end{matrix} \mapsto \begin{matrix} x_1^2 x_2 \\ x_1 x_2 \end{matrix}$ can be realized by homeo τ of torus  $\neq \Sigma_1^1$.
- Take $\Sigma_1^1 \times [0,1]$
(a genus two handlebody)



and glue on another handle to the annular part $\partial \Sigma_1^1 \times [0,1]$ of bdy.

- τ extends to homeo realising T
- A, B can be realised as <u>commuting</u> handleslides on top/bottom torus.



- Gives lower bound for DF. Upper bound via fellow-travelling surgery sequences in a geometric model for $\mathcal{H}_g$: define Φ^{nm}.
- Quadratic DF in genus 2: (define quasigeodesic combing) | Use special variant of pants graph (meridians + non-separating curves) and use intersection patterns in genus 2 to show it's a tree.

  Surgery paths exp. distorted

<u>Stabilisers (in progress)</u>

Another geometric difference between Out and Mcg:

In Mcg, stabilisers of curves / subsurfaces are always undistorted [Masur-Minsky]

In Out, situation is much more subtle [Handel-Mosher]

① ~~stab~~ free splitting stabilisers are undistorted

② free factor (of corank >1) stabilisers are exp. distorted.

This is how it should work for $\mathcal{H}_g$. (both Thm-in-progress)