The Heisenberg group has rational growth in all generating sets

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Abstract: Given a group G and a finite generating set $\mathcal{G}$ the (spherical) growth function $f_{\mathcal{G}}(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is the series whose coefficients an count the number of group elements at distance n from the identity in the Cayley graph $\Gamma_{\mathcal{G}}(G)$. For hyperbolic groups and virtually abelian groups, this is always the series of a rational function regardless of generating set. Many other groups are known to have rational growth in particular generating sets. In joint work with Moon Duchin, we show that the Heisenberg group also has rational growth in all generating sets. The first ingredient in this result is to compare the group metric, which we can see as a metric on the integer Heisenberg group with a metric on the real Heisenberg group. This latter is induced by a norm in the plane which is in turn induced by a projection of the generating set. The second ingredient is a wondrous theorem of Max Bensen regarding summing the values of polynomical over sets of lattice points in families of polytopes. We are able to bring these two ingredients together by showing that every group element has a geodesic whose projection into the plane fellow-travels a well-behaved set of polygonal paths.