Periodic orbit growth on covers of Anosov flows

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Anosov flows

M compact Riemannian manifold.

$\phi_t : M \to M$ (transitive) Anosov flow.

Topological entropy $h_{top}(\phi) > 0$.

Countably infinite set of prime periodic orbits $\mathcal{P}(\phi)$.

For $\gamma \in \mathcal{P}(\phi)$, write $\ell(\gamma)$ for its least period.

Exponential growth rate

For each $T > 0$, $\{\gamma \in \mathcal{P}(\phi) : \ell(\gamma) \leq T\}$ is finite. Furthermore,

$$\lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{P}(\phi) : \ell(\gamma) \leq T\} = h_{top}(\phi).$$

Covers

Let $\widetilde{M}$ be a regular cover of M with covering group G.

We can lift $\phi_t : M \to M$ to

$$\tilde{\phi}_t : \widetilde{M} \to \widetilde{M}.$$

We assume that the lifted flow is **transitive**.

Let $\mathcal{P}(\tilde{\phi})$ denote the set of prime periodic orbits for $\tilde{\phi}$. (It may be empty.)

We would like to understand the exponential growth rate associated to $\mathcal{P}(\tilde{\phi})$. (Note that some care is needed with the definition when G is infinite.)

Finite covers

If the covering group G is finite then $\widetilde{M}$ is compact and $\tilde{\phi}_t : \widetilde{M} \to \widetilde{M}$ is an Anosov flow.

Furthermore, $h_{top}(\tilde{\phi}) = h_{top}(\phi)$.

So
$$\lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{P}(\tilde{\phi}) : \ell(\gamma) \leq T\} = h_{top}(\phi).$$

Infinite covers

The question is more interesting when G is infinite.

First note that if $\gamma \in \mathcal{P}(\tilde{\phi})$ then, for each $g \in G$, the translate $g \cdot \gamma \in \mathcal{P}(\tilde{\phi})$ and $\ell(g \cdot \gamma) = \ell(\gamma)$. So, whenever $\{\gamma \in \mathcal{P}(\tilde{\phi}) : \ell(\gamma) \leq T\}$ is non-empty, it is also infinite.

Thus we need to restrict our counting.

Gurevič entropy

Following Paulin–Pollicott–Schapira, choose $K \subset \widetilde{M}$ open and relatively compact and define

$$h(\tilde{\phi}) := \lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{P}(\tilde{\phi}) : \ell(\gamma) \leq T, \ \gamma \cap K \neq \varnothing\}.$$

(The limit exists and is independent of the choice of K.)

We have $h(\tilde{\phi}) \leq h_{top}(\phi)$.

Question: When do we have equality?

Geodesic flows

A special class of Anosov flows are geodesic flows over a compact manifold with negative sectional curvatures:

$\phi_t : T^1N \to T^1N$, where N is a compact manifold with negative sectional curvatures.

These flows have *time-reversal symmetry*, i.e. there is a fixed point free involution $\iota : T^1N \to T^1N$ such that $\phi_t \circ \iota = \iota \circ \phi_{-t}$. (Here, ι is simply given by $\iota(x, v) = (x, -v)$, for $x \in N$ and $v \in T^1_x N$.)

This symmetry allows us to give a nice answer to the question in this case.

Geodesic flows

Let $M = T^1N$, where N is compact with negative sectional curvatures and let $\phi_t : M \to M$ be the geodesic flow.

Let $\widetilde{N}$ be a regular G-cover of N, let $\widetilde{M} = T^1\widetilde{N}$ and let $\tilde{\phi}_t : \widetilde{M} \to \widetilde{M}$ be the geodesic flow.

By a result of Eberlein, $\tilde{\phi}$ is transitive provided $G \neq \pi_1(N)$.

Theorem (Roblin, 2005; Dougall–S, 2016)

$h(\tilde{\phi}) = h_{top}(\phi)$ if and only if G is amenable.

Amenable groups

There are many ways of characterising amenable groups. A natural one is the following Følner criterion.

A countable group G is *amenable* if, for all finite sets $A \subset G$ and for all $\epsilon > 0$, there exists a finite set $F \subset G$ such that

$$\frac{\#(F \cap a \cdot F)}{\#F} > 1 - \epsilon$$

for all $a \in A$.

Symmetric extensions of shifts

A major ingredient of the proof is a theorem on Stadlbauer for group extensions of shifts. We only need it for subshifts of finite type but it also holds a large class of countable state shifts.

Let $\sigma : \Sigma \to \Sigma$ be a subshift of finite type, let G be a finitely generated group and let $\psi : \Sigma \to G$ be a function depending only on the first coordinate, $\psi((x_n))_{n=0}^{\infty}) = \psi(x_0)$. (This us just for simplicity – one can extend to any continuous ψ.)

Define a skew product $\sigma_\psi : \Sigma \times G \to \Sigma \times G$ by

$$\sigma_\psi(x, g) = (\sigma x, g\psi(x)).$$

Assume that σ_ψ is transitive.

Gurevič pressure

Let $f : \Sigma \to \mathbb{R}$ be Hölder continuous and define $\tilde{f} : \Sigma \times G \to \mathbb{R}$ by $\tilde{f}(x, g) = f(x)$.

The *Gurevič pressure* $P(\tilde{f}, \sigma_\psi)$ is defined by

$$P(\tilde{f}, \sigma_\psi) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} \exp\left(\sum_{i=0}^{n-1} f(\sigma^i x)\right),$$

where

$$\psi_n(x) = \psi(x)\psi(\sigma x) \cdots \psi(\sigma^{n-1} x)$$

and e is the identity in G. (This is a special case of Sarig's general definition.)

Stadlbauer's theorem

It is clear from the definition that

$$P(\tilde{f}, \sigma_\psi) \leq P(f, \sigma),$$

where $P(f, \sigma)$ is the standard pressure of f with respect to σ.

Theorem (Stadlbauer, 2013)

Suppose there is a fixed point-free involution $a \mapsto a^\dagger$ on the alphabet of Σ such that $\psi(a^\dagger) = \psi(a)^{-1}$ and that f satisfies a weak symmetry condition (which we omit). Then $P(\tilde{f}, \sigma_\psi) = P(f, \sigma)$ if and only if G is amenable.

Anosov flows

The amenability dichotomy for geodesic flows fails for general Anosov flows.

In fact, we can have G abelian with $h(\tilde{\phi}) < h_{top}(\phi)$.

This is easy if we drop transitivity for $\tilde{\phi} : \widetilde{M} \to \widetilde{M}$.

Mapping torus

Let $T : N \to N$ be an Anosov diffeomorphism and let

$$M = (N \times [0,1]) / \sim,$$

where $\sim$ is the identification

$$(x, 1) \sim (Tx, 0).$$

Then M is a compact manifold.

Now define $\phi : M \to M$ by

$$\phi_t(x, s) = (x, s + t) \text{ mod } \sim .$$

This is an Anosov flow.

$\mathbb{Z}$-cover of the mapping torus

M has a natural $\mathbb{Z}$-cover $\widetilde{M}$ with lifted flow $\tilde{\phi} : \widetilde{M} \to \widetilde{M}$.

It is easy to see that $\tilde{\phi}$ has no periodic orbits.

We can also have examples of transitive lifted flows on abelian covers where the growth rate drops.

Homology covers

Let $\phi_t : M \to M$ be a transitive Anosov flow.

We can consider the universal homology cover $\overline{M}$ and the lifted flow $\overline{\phi}_t : \overline{M} \to \overline{M}$. Here, the covering group is

$$H_1(M, \mathbb{Z}) \cong \mathbb{Z}^b \oplus (\text{torsion}),$$

where $b \geq 0$.

We will assume that $b \geq 1$ (so we have an infinite cover) and that $\overline{\phi}_t : \overline{M} \to \overline{M}$ is transitive. (The latter implies that $\phi_t : M \to M$ *homologically full*, i.e. that every homology class in $H_1(M, \mathbb{Z})$ is represented by a periodic orbit.)

We want to understand when $h(\overline{\phi})$ is equal to $h_{top}(\phi)$.

Winding cycles

We introduce the winding cycles (or asymptotic cycles) of Schwartzman.

Let μ be a ϕ-invariant probability measure on M. We define the associated *winding cycle*

$$\Phi_\mu \in H_1(M, \mathbb{R}) = H^1(M, \mathbb{R})^*$$

by

$$\langle \Phi_\mu, [\omega] \rangle = \int \omega(\mathcal{X}_\phi)\, d\mu,$$

where ω is a closed 1-form on M, $[\omega] \in H^1(M, \mathbb{R})$ is its cohomology class, $\mathcal{X}_\phi$ is the vector field generating ϕ, and

$$\langle \cdot, \cdot \rangle : H_1(M, \mathbb{R}) \times H^1(M, \mathbb{R}) \to \mathbb{R}$$

is the duality pairing. (Φ_μ is well-defined.)

Growth of null homologous periodic orbits

Let $\mathcal{M}_\phi$ denote the set of ϕ-invariant probability measures on M.

Theorem (S, 1993)

Suppose that $\phi_t : M \to M$ is homologically full. Then

$$h(\overline{\phi}) = \sup\{h_\phi(\mu) \, : \, \mu \in \mathcal{M}_\phi, \ \Phi_\mu = 0\}.$$

In fact, we have the precise asymptotic

$$\#\{\gamma \in \mathcal{P}(\phi) \, : \, \ell(\gamma) \leq T, \ [\gamma] = 0\} \sim C \frac{e^{h(\overline{\phi})T}}{T^{1+b/2}}, \text{ as } T \to \infty,$$

for some $C > 0$.

Growth of null homologous periodic orbits

Using the variational principle

$$h_{top}(\phi) = \sup\{h_\phi(\mu) : \mu \in \mathcal{M}_\mu\}$$

and the uniqueness of the measure of maximal entropy μ_0, we have:

Corollary

$h(\overline{\phi}) = h_{top}(\phi)$ if and only if $\Phi_{\mu_0} = 0$.

So if $\Phi_{\mu_0} \neq 0$ then $h(\overline{\phi}) < h_{top}(\phi)$ even though the covering group is amenable.

Result for general covers

What can we say about $h(\tilde{\phi})$ for a general regular G-cover $\widetilde{M}$ and the lifted flow $\tilde{\phi}_t : \widetilde{M} \to \widetilde{M}$?

Let $\overline{M}$ be the largest abelian subcover of $\widetilde{M}$, i.e. the cover of M with covering group $G/[G, G]$, the abelianisation of G.

Let $\overline{\phi}_t : \overline{M} \to \overline{M}$ be the lifted flow. Then $h(\overline{\phi})$ can be characterised in the same way as for the homology cover.

More precisely, we can define winding cycles $\Phi_\mu^{\overline{M}}$ relative to the cover $\overline{M}$ and

$$h(\overline{\phi}) = \sup\{h_\phi(\mu) : \mu \in \mathcal{M}_\phi, \ \Phi_\mu^{\overline{M}} = 0\}.$$

Result for general covers

It turns out that one should compare $h(\tilde{\phi})$ with $h(\overline{\phi})$.

Theorem (Dougall–S, 2019+)
$h(\tilde{\phi}) = h(\overline{\phi})$ if and only if G is amenable.

So the only mechanisms that allow $h(\tilde{\phi})$ to drop are

- G being non-amenable;
- "drift" in an abelian subcover.

Equidistribution

When we have an amenable cover, we can also describe the spatial distribution of periodic ϕ-orbits which lift to periodic $\tilde{\phi}$-orbits.

For $\xi \in ((G/[G,G]) \otimes \mathbb{R})^*$, let ω_ξ be a representative closed 1-form and let $F_\xi : M \to \mathbb{R}$ be the function

$$F_\xi(x) = \omega_\xi(\mathcal{X}_\phi(x)).$$

Let μ_ξ be the equilibrium state for F_ξ. Then the supremum

$$h(\overline{\phi}) = \sup\{h_\phi(\mu) : \mu \in \mathcal{M}_\phi, \ \Phi_\mu^{\overline{M}} = 0\}$$

is attained at μ_ξ.

Equidistribution

Associated to each periodic ϕ-orbit γ is a *Frobenius class* $\langle\gamma\rangle_G$ (a conjugacy class in G) so that γ lifts to a periodic $\tilde{\phi}$-orbit if and only if $\langle\gamma\rangle_G = \{e\}$. Let

$$\mathcal{P}_G(\phi) := \{\gamma \in \mathcal{P}(\phi) : \langle\gamma\rangle_G = \{e\}\}.$$

Write δ_γ for the Lebesgue measure on γ.

Theorem (Dougall–S, 2019+)

If G is amenable then

$$\frac{1}{\#\{\gamma \in \mathcal{P}_G(\phi) : \ell(\gamma) \leq T\}} \sum_{\substack{\gamma \in \mathcal{P}_G(\phi) \\ \ell(\gamma) \leq T}} \frac{\delta_\gamma}{\ell(\gamma)}$$

converges weak to μ_ξ, as $T \to \infty$.*

Equidistribution

We have $\xi = 0$ if and only if $\Phi_{\mu_0}^{\overline{M}} = 0$. This is satisfied by geodesic flows.

Corollary

For the geodesic flow over a compact negatively curved manilfold, if G is amenable then

$$\frac{1}{\#\{\gamma \in \mathcal{P}_G(\phi) : \ell(\gamma) \leq T\}} \sum_{\substack{\gamma \in \mathcal{P}_G(\phi) \\ \ell(\gamma) \leq T}} \frac{\delta_\gamma}{\ell(\gamma)}$$

converges weak to μ_0, as $T \to \infty$.*

Reference

R. Dougall and R. Sharp, Anosov flows, growth rates on covers and group extensions of subshifts, arXiv:1904.01423

Thank you for listening!