MODULAR CURVES AND THEIR ARITHMETIC 2023
PROBLEM SESSION

Q1. F. Najman

Let X be a curve defined over $\mathbb{Q}$ of genus ≥ 2 and $\mathbb{Q}$-gonality γ. Let $f : X \to \mathbb{P}^1$ be a degree d morphism. Suppose $f \neq g \circ h$ where $\deg(g), \deg(h) \geq 2$. Let

$$S = \{P \in f^{-1}(\mathbb{P}^1(\mathbb{Q})) \ : \ \deg(P) < d\}.$$

There are known instances for which S is always a finite set. For example when $d = 2$, Faltings' finiteness theorem asserts that S is finite. Suppose $J_X(\mathbb{Q})$ is finite where J_X is the Jacobian of X and suppose $d = \gamma$. Then S is a finite set.

Do there exist X and f for which S is an infinite set?

Q2. P. Parent

Let p be a prime. Recall that $X_0(p)$ is:

- a degree 2 cover of $X_0^+(p)$;
- a degree $p+1$ cover of $\mathbb{P}^1$;
- a degree $(p+1)/2$ cover of an elliptic curve of conductor p.

Is this a complete list of covered curves? In particular, is $X_0(p)$ a degree d cover of a curve X such that $d \geq (p+1)/3$?

Q3. R. Visser

Does there exists a smooth genus 2 curve $C/\mathbb{Q}$ such that $\mathbb{Q}(J[2]) = \mathbb{Q}$ and $\mathbb{Q}(J[4]) = \mathbb{Q}(i)$, where J is the Jacobian of C?

C. Maistrét: Use Richelot isogeny to try to prove that this is never true. H. Yoo: Consider taking the product of 2 elliptic curves.

Q4. B. Bhatta

Let V be a $M_d(\mathbb{C})$ module. Let $\phi : V \times V \to M_d(\mathbb{C})$ be a non-degenerate skew Hermitian form. Does there exist $z \in V$ such that $\phi(z,z) \in \mathrm{GL}_d(\mathbb{C})$?

Q5. D. K. Angdinata

Let E be an elliptic curve over $\mathbb{Q}$. Let $\phi : X_0(N) \to E$ denote the modular parameterisation. Let c_0 denote the Manin constant. Cremona conjectured that $c_0 \leq 5$. Suppose that the following hold:

- $L(E,1) \neq 0$;
- $3 \nmid c_0$;
- $im(\rho_{E,3}) = 9.24.02$.

Why does 9 divide the Tamagawa number of E?

Date: December 10, 2023.

Q6. C. Maistrét

Let C be a fixed Frey hyperelliptic curve. Write $C : y^2 = f(x)$. Is it true that $\mathrm{Gal}(f)$ is necessarily a product of cyclic groups? Is is true that $\mathrm{Gal}(f)$ is necessarily solvable? If so, can this be proved from the definition of C? (For the 3 known curves C, the answer is yes.)

Q7. S. Anni

Let E be an elliptic curve over $\mathbb{Q}$. Let $[E]$ denote the isogeny class of E. Kenku showed that $\#[E] \leq 8$. Let $p > 37$ be a prime. If E has conductor p such that $p \nmid (6 + u^2 + v^2)$ then a result of Serre asserts that $\#[E] = 1$. Is there a uniform bound if E is defined over a number field and has prime conductor norm? For example, if E is defined over $\mathbb{Q}(\sqrt{29})$?