Comparing standard and conjugacy growth in infinite groups

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LMS Summer School on Growth and Expansion in Groups

7th of July, 2026

Finite groups

Let S_3 be the symmetric group on 3 objects.

$$S_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$$

- Elements : words on generating set $\{a, b\}$:

$$\{1, a, b, ab, ba, aba\}$$

  How many elements? 6

  Generating function?

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$$\texttt{elements of length} = n: \qquad s(n) := \sharp\{g \in G \mid |g|_X = n\}$$

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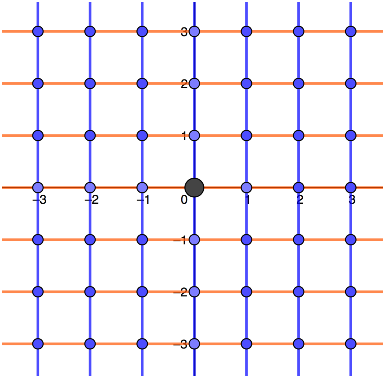
- Conjugacy growth functions:

$$c(n) := \sharp\{[g] \in G \mid |g|_c = n\}$$

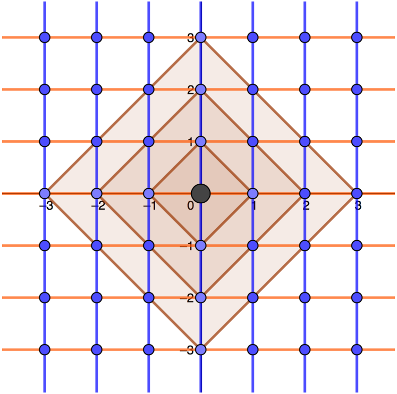
$$C(n) := \sharp\{[g] \in G \mid |g|_c \leq n\}$$

Examples: free groups – abelian and non-abelian

Cayley graph of $\mathbb{Z}^2$ with standard generators a and b



$\mathbb{Z}^2$ with standard generators a and b



$$s(k) = 4k, \quad S(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

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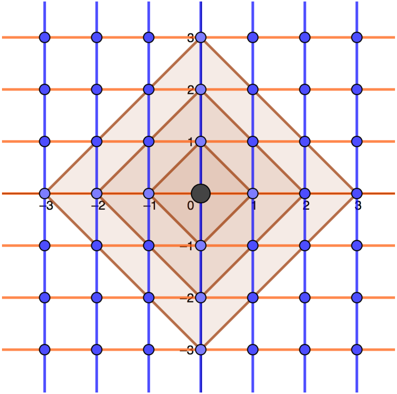
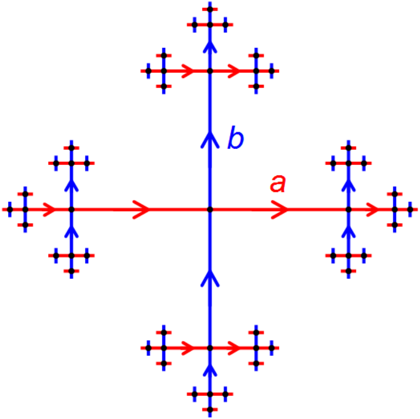


$$s(k) = c(k) = 4k, \quad S(n) = C(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

Free group $F(a, b)$ with generators a and b



$$s(n) = 4 \cdot 3^{n-1}$$

Conjugacy growth in $F(a, b)$?

- $[aba] = \{aba, baa, aab, a^3ba^{-1}, \dots\}$

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Not entirely correct: when powers are included, one shouldn't divide by n.

Asymptotics of conjugacy growth in free groups

Coornaert (2005): For the free group F_r, the primitive (non-powers) conjugacy growth function is given by

$$c_p(n) \sim \frac{(2r-1)^{n+1}}{2(r-1)n} = K\,\frac{(2r-1)^n}{n},$$

where $K = \frac{2r-1}{2(r-1)}$.

Comparing standard and conjugacy growth

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  $\star$ Exclude the Osin or Ivanov type 'monsters'!

- Easy/Hard: Compare standard and conjugacy growth rates.

Growth rates

The standard growth rate $\limsup_{n\to\infty} \sqrt[n]{s(n)}$ of G wrt X is in fact a limit i.e.

$$\alpha = \lim_{n\to\infty} \sqrt[n]{s(n)}.$$

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Hull: There are groups for which

$$\liminf_{n\to\infty} \sqrt[n]{c(n)} < \limsup_{n\to\infty} \sqrt[n]{c(n)},$$

that is, the limit does not exist.

Conjugacy growth asymptotics and the Guba - Sapir conjecture

- Breuillard-Cornulier (2010):

  exponential conjugacy growth for soluble (non virt. nilpotent) groups.

- Breuillard-Cornulier-Lubotzky-Meiri (2011):

  exponential conjugacy growth for linear (non virt. nilpotent) groups.

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  exponential conjugacy growth for linear (non virt. nilpotent) groups.

- Hull-Osin (2014):

  acylindrically hyperbolic groups have exponential conjugacy growth.

Conjugacy vs. standard growth

	Standard growth	Conjugacy growth
Type	pol., int., exp.	pol., int.*, exp.
Quasi-isometry invariant		
Rate of growth		

* Bartholdi, Bondarenko, Fink.

Conjugacy vs. standard growth

	Standard growth	Conjugacy growth
Type	pol., int., exp.	pol., int.*, exp.
Quasi - isometry invariant	yes	no**, but group invariant
Rate of growth	exists	exists (not always)

* Bartholdi, Bondarenko, Fink.

** Hull-Osin (2013): conjugacy growth not quasi-isometry invariant.

1st Recap

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  'Robust' asymptotics.

- growth of conjugacy classes.

  A few dozen papers.

  'Less robust' asymptotics.

Conjugacy growth: history and motivation

Conjugacy growth in geometry

Counting the primitive closed geodesics of bounded length on a compact manifold M of negative curvature and exponential volume growth gives

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Counting the primitive closed geodesics of bounded length on a compact manifold M of negative curvature and exponential volume growth gives

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- 1960s (Sinai, Margulis):
  $M =$ complete Riemannian manifolds of pinched negative curvature.
- 1990s - 2000s (Knieper, Coornaert, Link):
  some classes of (rel) hyperbolic or CAT(0) groups.

Conjugacy growth asymptotics

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- Rivin (2000), Coornaert (2005):

  asymptotics for the free groups.

- Guba-Sapir (2010):

  asymptotics for various (non-geometric) groups.

Growth series

The standard growth series

Let G be a group with finite generating set X.

- The standard growth series of G with respect to X records the number of elements of every length. It is

$$\sigma_{(G,X)}(z) := \sum_{n=0}^{\infty} s(n) z^n,$$

where $s(n)$ is the number of elements of length n.

Growth series for $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z} = \langle a \mid \ \rangle$ the standard growth series is:

$$\sigma_{(\mathbb{Z},\{a,a^{-1}\})}(z) = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

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In $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$ the elements are $1, a, b, ab, ba,$ $aba, bab \ldots$, so the standard growth series is:

$$\sigma_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + \ \cdots = \frac{??}{??}.$$

The conjugacy growth series

Let G be a group with finite generating set X.

- The conjugacy growth series of G with respect to X records the number of conjugacy classes of every length. It is

$$\widetilde{\sigma}_{(G,X)}(z) := \sum_{n=0}^{\infty} c(n) z^n,$$

where $c(n)$ is the number of conjugacy classes of length n.

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ the representatives for conjugacy classes are $1, a, b, ab, \ldots,$ so

$$\widetilde{\sigma}_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + \ \cdots = \frac{??}{??}.$$

Rational, algebraic, transcendental

A generating function $f(z)$ is

- rational if there exist polynomials $P(z)$, $Q(z)$ with integer coefficients such that $f(z) = \frac{P(z)}{Q(z)}$;
- algebraic if there exists a polynomial $P(x, y)$ with integer coefficients such that $P(z, f(z)) = 0$;
- transcendental otherwise.

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- transcendental otherwise.

Remark: If $f(z) = \frac{P(z)}{Q(z)}$ is a rational generating function for some sequence a_n, the roots of $Q(z)$ give the growth rate of a_n.

2nd Recap: main questions about (conjugacy) growth

- What are the asymptotics of the (conjugacy) growth of a group?
- When is the (conjugacy) growth series of a group a rational function?
  Algebraic?
  Transcendental?
- How do the algebraic and geometric properties of a group influence the complexity of the growth (series)?

Growth series in groups: rationality

Groups with lots of commuting elements (i.e. positive curvature)

Virtually Abelian Groups

Examples

$\mathbb{Z}^n$, affine Coxeter groups, Bieberbach groups.

Theorem [Benson, Invent. Math., 1983]

Every virtually abelian group has rational standard growth series with respect to any choice of finite generating set.

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Theorem [Evetts, 2019]

Every virtually abelian group has rational conjugacy growth series with respect to any choice of finite generating set.

Rationality

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Theorem [Stoll, 1996]

The higher Heisenberg groups H_r, $r \geq 2$, have rational growth with respect to one choice of generating set and transcendental with respect to another.

$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c, d, e \in \mathbb{Z} \right\}$$

Groups of polynomial standard growth: some tools

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Lemma [Stoll 1996]

If $a(n) \sim \alpha n^d$ and α is a irrational (resp. transcendental) number, then the corresponding power series $\sum a(n)z^n$ is irrational (resp. transcendental).

Conjugacy growth in Heisenberg groups

Conjugacy growth in Heisenberg group(s) [Evetts '22]

Definition

The discrete Heisenberg group $H_1 = \langle a, b \mid [a, b] \text{ central} \rangle$.

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- If $(i, j) \neq (0, 0)$, then $[a^i b^j z^k] = a^i b^j z^k \langle z^{\gcd(i,j)}\rangle$.

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- So the conjugacy growth series cannot be rational, by the result of Stanley.

Heisenberg group(s): conjugacy growth

Proposition

The conjugacy growth of the first Heisenberg group (3×3 matrices) is $\sim n^2 \log(n)$.[1]

[1] for all generating sets

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Theorem [Bodart, Evetts, 2023]

The conjugacy growth series for the higher Heisenberg groups is transcendental for the standard generating sets.

[1] for all generating sets

Overview for groups of positive curvature

	Standard Growth Series	Conjugacy Growth Series
Virtually abelian	Rational (Benson '83)	Rational (Evetts '19)
Heisenberg H_1	Rational (Duchin-Shapiro '19)	Transcendental

FOR ALL GENERATING SETS!

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FOR ALL GENERATING SETS!

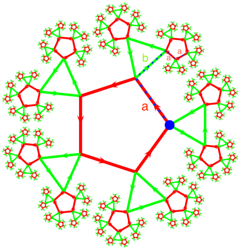
Lots of other results known for further nilpotent, soluble, metabelian etc groups, but mostly for specific generating sets.

Groups with few commuting elements (i.e. negative curvature)

Hyperbolic groups

Motivation: Most (finitely presented) groups are hyperbolic.

'Definition': Groups whose Cayley graph looks like the hyperbolic plane.



Examples: free groups, free products of finite groups, $SL(2, \mathbb{Z})$, virtually free groups *, surface groups, small cancellation groups, and many more.

* Virtually free = groups with a free subgroup of finite index.

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Proof ideas:

Conjugacy growth in hyperbolic groups ???

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F_k = free group on k generators

- $\widetilde{\sigma}(z) :=$ the conjugacy growth series of F_k wrt standard generators
- Rivin (2000, 2010): the conjugacy growth series of F_k is not rational!

$$\widetilde{\sigma}(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

Conjecture (Rivin, 2000)

If G hyperbolic, then the conjugacy growth series of G is rational if and only if G is virtually cyclic (i.e. $\mathbb{Z}$ or virtually $\mathbb{Z}$).

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$\Rightarrow$

Theorem (Antolín - C., 2017)

If G is non-elementary hyperbolic, the conjugacy growth series is transcendental.

Conjecture (Rivin, 2000)

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$\Leftarrow$

Theorem (C. - Hermiller - Holt - Rees, 2016)

If G is virtually cyclic, the conjugacy growth series of G is rational.

NB: Both results hold for all generating sets of G.

Theorem (Antolín - C., 2017)

If G is hyperbolic[1], the conjugacy growth series is transcendental.

[1] not virtually cyclic

Idea of proof: GEOMETRY

Theorem. (Coornaert - Knieper 2007, Antolín - C. 2017)

Let G be a 'large' hyperbolic group. There are constants $A, B, n_0 > 0$ such that

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

for all $n \geq n_0$, where α is the word growth rate of G.

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MESSAGE:

The number of conjugacy classes of length n is asymptotically the number of elements of length n divided by n.

End of the proof: COMBINATORICS

The transcendence of the conjugacy growth series follows from the bounds

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End of the proof: COMBINATORICS

The transcendence of the conjugacy growth series follows from the bounds

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

together with

Lemma (Flajolet: Trancendence of series based on bounds).

Suppose there are positive constants $A, B, \mathbf{h}$ and an integer $n_0 \geq 0$ s.t.

$$A\frac{e^{\mathbf{h}n}}{n} \leq a_n \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$. Then the power series $\sum_{i=0}^{\infty} a_n z^n$ is not algebraic.

3rd Recap: growth in hyperbolic groups

- The standard growth series of a hyperbolic group is rational for all gen. sets.
- The conjugacy growth series of a hyp. group is transcendental for all gen. sets.

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- The standard growth series of a hyperbolic group is rational for all gen. sets.
- The conjugacy growth series of a hyp. group is transcendental for all gen. sets.
- **Corollary** (Antolín - C.)

  For any hyperbolic group G with generating set X the standard and conjugacy growth rates are the same:

$$\lim_{n\to\infty} \sqrt[n]{c(n)} = \gamma_{G,X} = \alpha_{G,X}.$$

Rivin's conjecture for other groups: GEOMETRY

Theorem (Gekhtman and Yang, 2022)

Let G be a non-elementary group with a finite generating set S.

If G has a contracting element with respect to the action on the corresponding Cayley graph,

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Why is the conjugacy growth series transcendental?

Groups with contracting element action on Cayley graph, GY '22

- Relatively hyperbolic groups,
- right-angled Coxeter groups,
- right-angled Artin groups,
- graphical small cancellation groups.

All have transcendental conjugacy growth series w.r.t. standard generating sets.

Careful with contracting elements!

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Careful with contracting elements!

- There are dozens of papers finding contracting elements for group actions, but ...

  they do not give any results about conjugacy growth because the group action is not on the Cayley graph.

- There are very few known examples where the contracting elements are w.r.t. action on Cayley graph.

Groups with useful contracting elements

Coxeter groups (C. & Genevois, 2025)

Theorem

Let (W, S) be an irreducible non-affine Coxeter group (i.e. not virtually abelian).

Then there is a contracting element for the action of W on the Cay(W, S).

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$$\Longrightarrow$$

Corollary

Coxeter groups have transcendental conjugacy growth series with respect to their standard generating sets.*

More groups and their growth series w.r.t. standard gen. sets

	Conjugacy Series	Standard Series
Certain wreath products	Transcendental (Mercier '17)	rational/algebraic
Graph products[2]	Transcendental (C. - Genevois '25) (C.- Hermiler - Mercier '23, '26)	rational
BS(1,m)	Transcendental (C.- Evetts - Ho, '20)	rational
Nilpotent[3]	Transcendental (Evetts '23)	rational transcendental
Dihedral Artin groups	Transcendental (C.-Crowe, '25)	rational

[2] based on series for vertex groups
[3] certain virtually nilpotent (class 2)

Final recap

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- Conjugacy growth series for all non - virtually abelian groups studied so far are transcendental.
- Source of transcendental-ness: counting cyclic representatives of a set introduces Euler's φ, infinitely many poles etc.
- **Conjecture (C, Evetts, Ho)**:
  The only finitely presented groups with rational conjugacy growth series are virtually abelian.

Questions

- Are the standard and conjugacy growth rates equal for all ‘natural’ groups?

  This holds for EVERY class of groups studied so far, but the proof is local.

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  This holds for EVERY class of groups studied so far, but the proof is local.

- Are there groups with algebraic conjugacy growth series?

- How do the conjugacy growth series behave when we change generators?

  Stoll: The rationality of the standard growth series depends on generators.

Thank you!

Computing the conjugacy growth series

Counting cyclic representatives

- Let L be a set of words, $a(k)$ the number of words of length k in L, and $f_L(t) = \sum_{k \geq 1} a(k) t^k$ the generating function of L.

  Assume L is closed under taking powers and cyclic permutations of words.

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  Assume L is closed under taking powers and cyclic permutations of words.

- The generating function for the language $L_\sim$ of cyclic representatives is

$$\boxed{\int_0^z \frac{\sum_{k \geq 1} \phi(k) f_L(t^k)}{t} \, dt.}$$

Rivin's formula for free groups

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- Rivin's formula for conjugacy series of free groups

$$\widetilde{\sigma}(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

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- SURPRISE:

$$f_{L_\sim}(z) = \widetilde{\sigma}(z)$$