Toric decomposition in algebraic groups

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Outline

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Toric packing/covering number

Setup: G connected reductive algebraic group, $T_1, \ldots, T_s$ maximal tori.
e.g. $G = \mathrm{PGL}_n(\mathbb{F}) = \mathrm{GL}_n(\mathbb{F})/\mathbb{F}^\times$, $T_i \cong \{\text{diagonal matrices}\}/\mathbb{F}^\times$.

Problem

$\mu\colon T_1 \times \cdots \times T_s \to G$, $(t_1, \ldots, t_s) \mapsto t_1 \ldots t_s$ *the multiplication map.*

- $\textsc{Pack}(G) = \max\{s : \mu \text{ injective}\} =?$ $(T_1 t_2 \ldots t_s : t_i \in T_i)$
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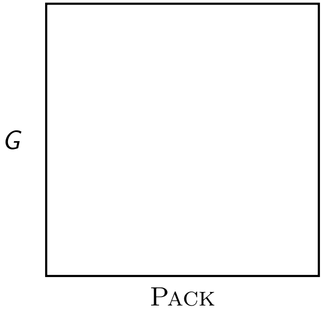

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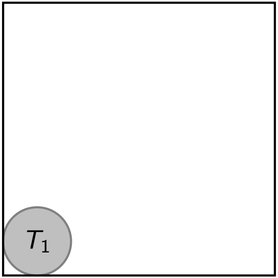

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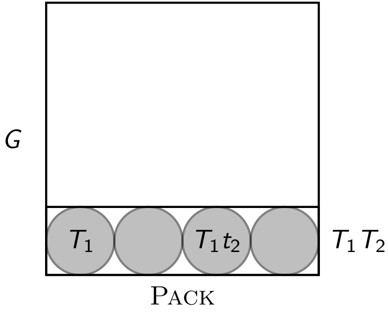

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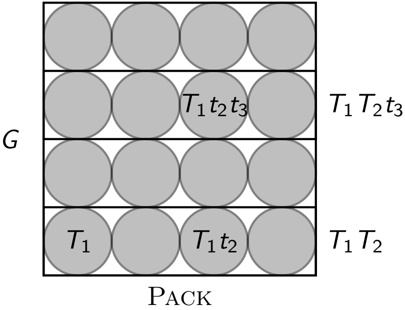

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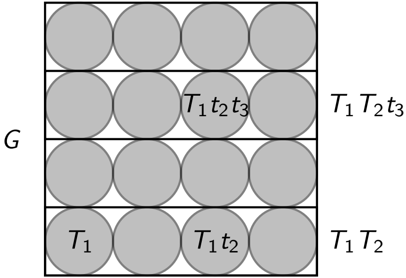

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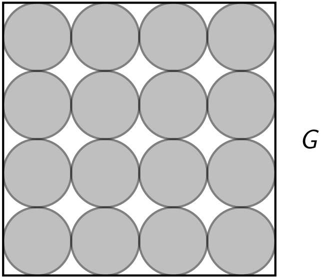

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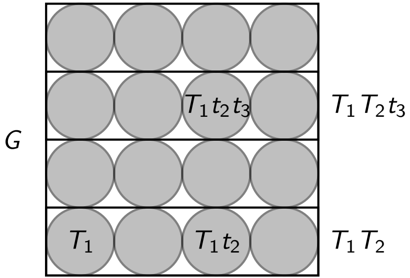

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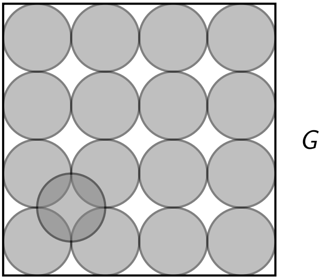

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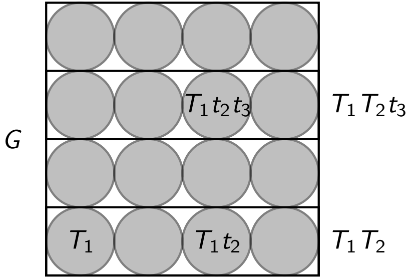

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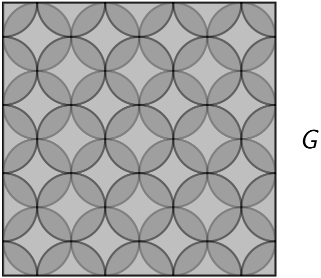

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$$1 \le \textsc{Pack}(G) \overset{*}{\le} \frac{\dim(G)}{\mathrm{rank}(G)}, \quad \frac{\dim(G)}{\mathrm{rank}(G)} \overset{\dagger}{\le} \textsc{Cover}(G) \le 2\dim(G)$$

$$\textsc{Pack}(\mathrm{PGL}_n) \overset{*}{=} \frac{\dim}{\mathrm{rank}} = n+1, \qquad \textsc{Cover}(\mathrm{PGL}_n) \overset{\dagger}{\in} \{n+1, n+2\}$$

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Proof – Part 1: connected reductive G

Theorem (Recall: Part 1)

G connected, reductive: $\textsc{Pack}(G) \overset{(1)}{\leq} \frac{\dim(G)}{\operatorname{rank}(G)} \overset{(2)}{\leq} \textsc{Cover}(G) \overset{(3)}{\leq} 2\dim(G)$

Proof:

- (1): Trivial: packing $T_1 \times \cdots \times T_s \hookrightarrow G$, $\dim(T_i) = \operatorname{rank}(G)$
- (2): Trivial: covering $T_1 \times \cdots \times T_s \twoheadrightarrow G$
- (3): **Construction**:
  - **Growth**: $\forall V \subsetneq G$ subvariety, $\exists T$ maximal torus:

    $$\dim(V) \lneq \dim(VT)$$

    (because $\bigcup$ tori $= \{\text{semisimple}\} \subseteq G$ dense in connected reductive)
  - **Dense**: Increasing chain (at most $\dim(G) - \operatorname{rank}(G) + 1$ steps):

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  - **Density→cover**: If $U_1, U_2 \subseteq G$ nonempty open, then $U_1 U_2 = G$.

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Proof – Part 2: packing construction in PGL_n

Theorem (Recall: Part 2)

V is n-dimensional $\mathbb{F}$-vector space, $n > 1$, $\mathbb{F}$ arbitrary.
$\exists T_n, \ldots, T_0 \subset \mathrm{PGL}(V)$ maximal tori: $\mu\colon T_n \times \cdots \times T_0 \hookrightarrow \mathrm{PGL}(V)$.

Setup:

- $\mathrm{PGL}(V)$ acts on $\mathbb{P}(V)$
- maximal torus = pointwise stabiliser of n independent points
- $\mathrm{PGL}(V)$ acts regularly on **frames** ($(n+1)$-tuples in general position)

Proof:

- Fix frame $(P_0, \ldots, P_n)$, define max. torus $T_i = \mathrm{Stab}(P_j : j \neq i)$.
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- maximal torus = pointwise stabiliser of n independent points
- $\mathrm{PGL}(V)$ acts regularly on **frames** ($(n+1)$-tuples in general position)

Proof:

- Fix frame $(P_0, \ldots, P_n)$, define max. torus $T_i = \mathrm{Stab}(P_j : j \neq i)$.
- $T_i \ldots T_0 \subseteq H_i := \mathrm{Stab}(P_j : j > i)$, stabiliser of $n - i$ points
- Injectivity: $T_{i+1} \cap (T_i \ldots T_0) \subseteq T_{i+1} \cap H_i = \mathrm{Stab}(P_0, \ldots, P_n) = 1$

Proof – Part 2: packing construction in PGL_n

Theorem (Recall: Part 2)

V is n-dimensional $\mathbb{F}$-vector space, $n > 1$, $\mathbb{F}$ arbitrary.
$\exists T_n, \ldots, T_0 \subset \mathrm{PGL}(V)$ *maximal tori:* $\mu \colon T_n \times \cdots \times T_0 \hookrightarrow \mathrm{PGL}(V)$.

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Proof – Part 3: covering construction in PGL_n

Theorem (Recall: Part 3)

V n-dim. $\mathbb{F}$-vector space ($\mathbb{F}$ arbitrary with $|\mathbb{F}| > n^2$),
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- **Good frames** : $\{Q : \overbrace{(Q_0, \ldots, Q_n)}^{Q} \xrightarrow{g \in T_n \ldots T_0} \overbrace{(P_0, \ldots, P_n)}^{\text{fixed frame}}\}$ is **open**:
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Proof – Part 3: covering construction in PGL_n

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$$1 \overset{T_0}{\lnsim} H_0 \overset{T_1}{\lnsim} H_1 \overset{T_2}{\lnsim} \dots \overset{T_{n-1}}{\lnsim} H_{n-1} \overset{T_n}{\lnsim} \mathrm{PGL}(V)$$

- **Good frames** : $\{Q : \overbrace{(Q_0, \dots, Q_n)}^{Q} \xrightarrow{g \in T_n \dots T_0} \overbrace{(P_0, \dots, P_n)}^{\text{fixed frame}}\}$ is **open**:
  $(Q_0, \dots, Q_n) \xrightarrow{\exists t_0} (P_0, Q_1', \dots, Q_n') \xrightarrow{\exists t_1} (P_0, P_1, Q_2'' \dots, Q_n'') \xrightarrow{\exists t_2}$
  $(P_0, P_1, P_2, Q_3''' \dots, Q_n''') \xrightarrow{\exists t_3} \dots \xrightarrow{\exists t_n} (P_0, \dots, P_n)$
  Open: e.g. for $\exists t_0$, need $Q_0, P_1, \dots, P_n$ be in general position
- **Bad frames**: $X_1, \dots, X_{n^2}$ hypersurfaces (in space of frames $\cong \mathrm{PGL}_n$)
- **Escape** bad frames in one step: 'bad frame' $\xrightarrow{t \in T}$ 'good frame'
  - Need: one maximal torus $T = \mathrm{Stab}(R_1, \dots, R_n)$: $\forall T$-orbit $\not\subseteq \bigcup_i X_i$
  - Combinatorial Nullstellensatz + computation:
    Works for $\{P_0, \dots, P_n, R_1, \dots, R_n\}$ general position and $|\mathbb{F}| > n^2$

Outline

Conjectures

G simple algebraic (or even reductive?) group:

- $C_1 \frac{\dim(G)}{\mathrm{rank}(G)} \leq \mathrm{PACK}(G) \leq \frac{\dim(G)}{\mathrm{rank}(G)} \leq \mathrm{COVER}(G) \leq C_2 \frac{\dim(G)}{\mathrm{rank}(G)}$.
- Torus growth: $\exists T_1, \dots, T_s$ max. tori for all $s \leq \frac{\dim(G)}{\mathrm{rank}(G)}$:
$$\dim(T_1 \dots T_s) = s \cdot \mathrm{rank}(G)$$
- More generally: $\forall V \subseteq G$ closed, $\exists$ max. torus T:
$$\dim(VT) = \min\{\dim(V) + \dim(T), \dim(G)\}$$

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Thank you for your attention!

[Sza26] Dávid R. Szabó. *Toric decomposition in algebraic groups*. 2026. arXiv: `2607.05212 [math.AG, math.GR]`.