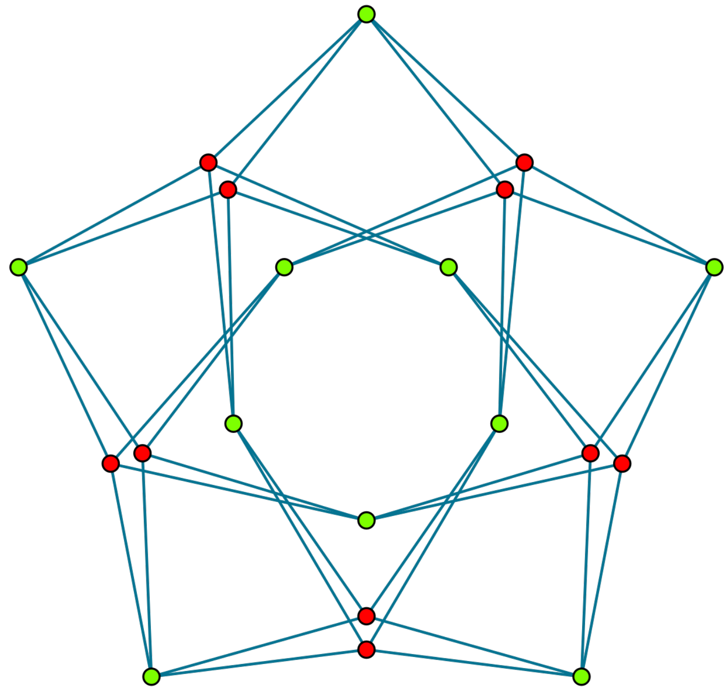
Fixed-point-free elements in permutation groups with two orbits

Joint work with Sean Eberhard



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LMS Summer school on growth and expansion in groups
University of Warwick, 6 July 2026

Derangements

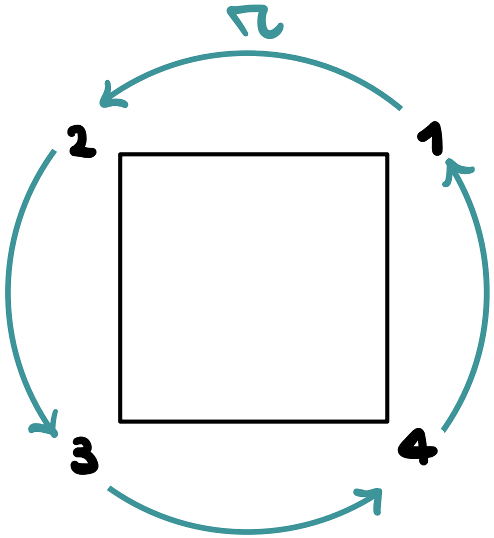
Ω finite set

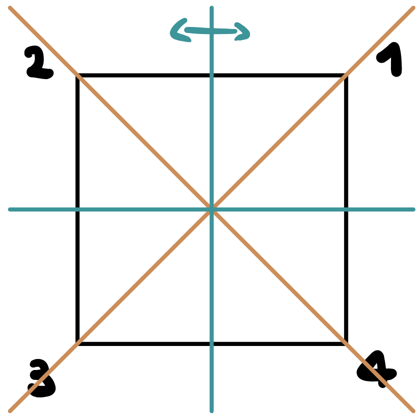
$G \leq \mathrm{Sym}(\Omega)$

An element $g \in G$ is a **DERANGEMENT** if it has no fixed points on Ω, i.e.

$$\alpha g \neq \alpha \quad \forall\, \alpha \in \Omega$$

- $G = D_8 = \langle \pi, s \mid \pi^4, s^2, s\pi s = \pi^{-1} \rangle \quad \pi = (1234) \quad s = (13)$





$$\left.\begin{aligned} \pi &= \rho_{\frac{\pi}{2}} = (1234) \\ \pi^2 &= (13)(24) \\ \pi^3 &= (1432) \end{aligned}\right\}$$ All non-trivial rotations are derangements

(13), (24) are NOT derangements

(12)(34), (14)(23) are derangements

Transitive groups

THM (JORDAN, 1872): A transitive permutation group on a finite set of size $n>1$ contains a derangement.

Equivalently, if $H = G_\alpha$, $\bigcup_{g \in G} H^g \subsetneq G$.

On a theorem of Jordan, J-P. Serre, 2003.

↳ Jordan's theorem has interesting implications in number theory and topology.

Transitive groups

THM (JORDAN, 1872):

A transitive permutation group on a finite set of size $n>1$ contains a derangement.

Proof: Just apply the ORBIT-COUNTING LEMMA.

THM (FEIN-KANTOR-SCHACHER, 1981):

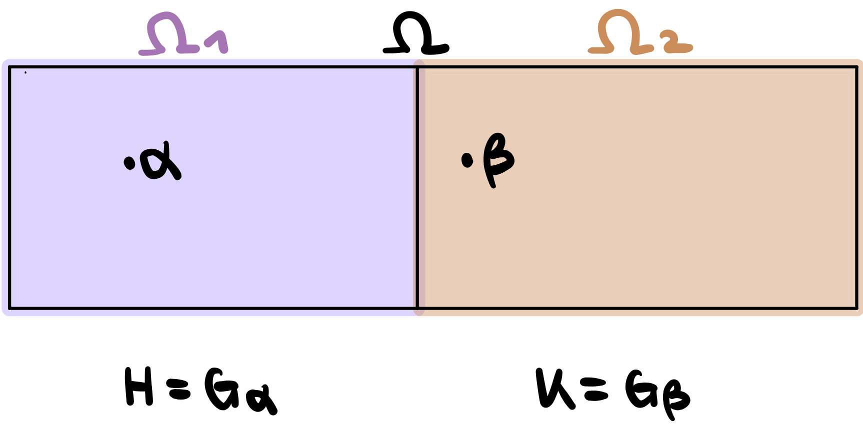
A transitive permutation group on a finite set of size $n>1$ contains a derangement of prime-power order.

Proof: Requires the CFSG.

Permutation groups with two orbits

Assume $G \leq \mathrm{Sym}(\Omega)$ has two orbits : $\Omega = \Omega_1 \sqcup \Omega_2$

When does G have a derangement ?



When is $\bigcup_{g \in G} H^g \cup \bigcup_{g \in G} K^g \subsetneq G$?

Assume $G \leq \mathrm{Sym}(\Omega)$ has two orbits : $\Omega = \Omega_1 \sqcup \Omega_2$

Received: 29 September 2025 | Revised: 26 January 2026 | Accepted: 26 January 2026

DOI: 10.1112/jlms.70457

RESEARCH ARTICLE

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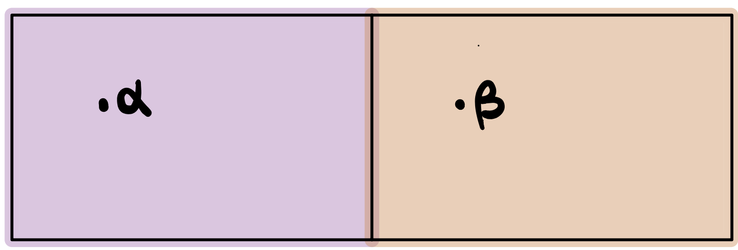
Derangements in intransitive groups

David Ellis[1] | Scott Harper[2]

Conjecture 1. *Let $G \leqslant \mathrm{Sym}(n)$ have two orbits of size $\frac{n}{2} > 1$. Then G contains a derangement.*

$|\Omega_1| = |\Omega_2|$



$H = G_\alpha$ $K = G_\beta$

$|H| = |K|$

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↳ They prove the conjecture if G is soluble, almost simple, primitive on one of the two orbits, ...

→ Further results were obtained by Lee-Popiel-Verret

THM (A., Eberhard, '26$^+$): Let $G \leq \mathrm{Sym}(\Omega)$ be a two-orbit permutation group with $|\Omega| > 2$. Then G contains either

- a derangement, or
- an element of prime-power order with a unique fixed point.

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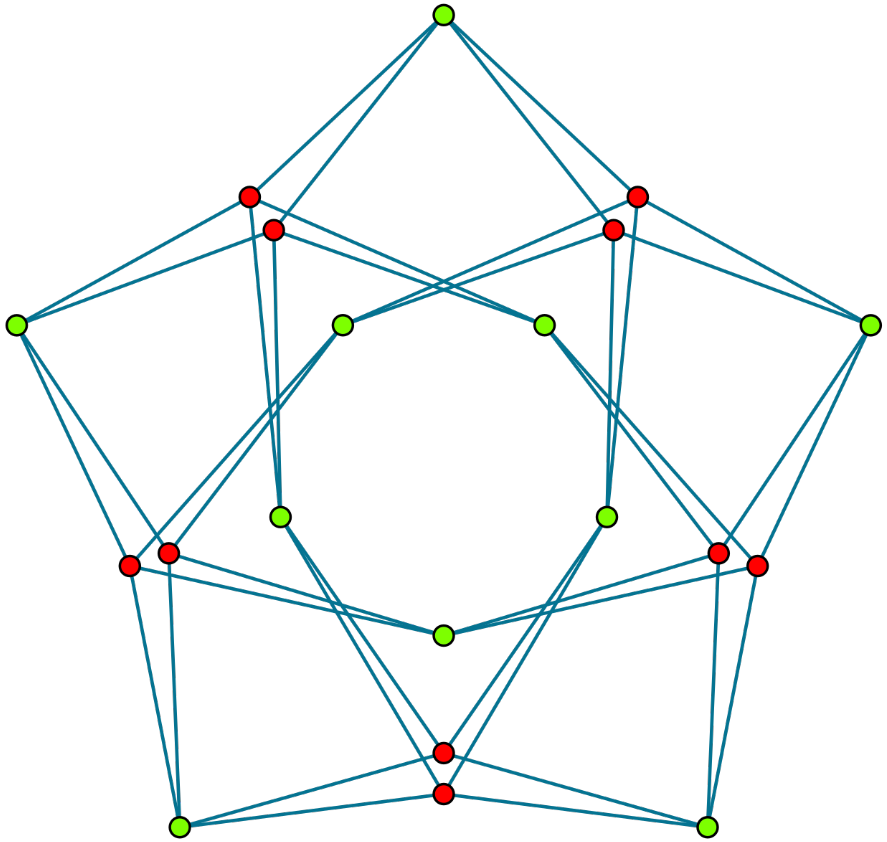
COROLLARY: Let $G \le \mathrm{Sym}(\Omega)$ be a two-orbit permutation group with orbits of size n_1 and n_2, $n = n_1 + n_2 > 2$.
If $\gcd(n_1, n_2 - 1) = \gcd(n_1 - 1, n_2) = 1$, then G contains a derangement.

→ In particular, the Ellis-Harper conjecture holds true.

Example:

Folkman graph



COROLLARY: If Γ is a finite edge-transitive d-regular graph, $d>1$, then $\mathrm{Aut}(\Gamma)$ has a derangement on $V(\Gamma)$.

Thank you!