Cartan subgroups and Strong Approximate Lattices

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Cartan subgroups

Cartan subgroup: A subgroup C of a Lie group G is called a Cartan subgroup if it satisfies the following properties:

(i) C is a maximal nilpotent subgroup of G.

(ii) if L is any closed normal subgroup of finite index in C, then L has finite index in its own normalizer in G.

Examples: (1) Cartan subgroup of a connected nilpotent group is itself.
(2) For $G = \mathbb{R} \ltimes \mathbb{R}^2$, where $t \cdot v = e^{it}v$, every Cartan subgroup is conjugate to $\mathbb{R} \times \{0\}$.
(3) If G is connected and compact, then the Cartan subgroups are precisely the maximal tori.
(4) For $\mathrm{SL}(2, \mathbb{R})$ there are two 'type' of Cartan subgroups, namely: $\mathrm{SO}(2, \mathbb{R})$ and $\left\{(a, a^{-1}) : a \in \mathbb{R}^\times\right\}$.

- Cartan subgroups are centraliser of its maximal tori in an algebraic group. For a semisimple algebraic group, they are the same as maximal tori.
- There is a one-one correspondence between Cartan subgroups and Cartan subalgebras (Chevalley).

Motivation

Cartan subgroups have been extensively studied by Chevalley, Borel, Togo, Prasad–Raghunathan, Hofmann, Goto, Neeb, Wüstner, Winkelmann, Chatterjee, Dani, Shah, to name a few.

- A Waring-Type Problem for Word Maps: For a connected Lie group G, any element $g \in G$ can be written as $g = a^k b^k$ for any $k \geq 2$. In particular $G = P_k(G)^2$ (Wüstner).
- Density of the images of Power Maps: Let G be a semisimple Lie group. Then $P_k(G)$ is dense in G if and only if $P_k(C) = C$ for every Cartan subgroup C of G (Bhaumik and Mandal).
- Connection with Lattices for Determining the Real Rank of Semisimple Lie Groups: Let G be a semisimple Lie group without compact factor. They showed, Lattices and Cartan subgroup intersect in a nice way. It helps to characterise the real rank of G (Prasad and Raghunathan).

Work of Prasad–Raghunathan

Theorem: Let G be a semisimple Lie group without a compact factor. Let Γ be a Lattice in G and C be a non compact Cartan subgroup of G. Then $gCg^{-1} \cap \Gamma$ contains a regular element for some $g \in G$. Moreover, $gCg^{-1} \cap \Gamma$ is a uniform Lattice in gCg^{-1}.

Theorem: Let G_1 and G_2 be connected semisimple Lie groups without compact factors. Let $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ be lattices and they are isomorphic as an abstract group, then $\mathbb{R}$-rank$(G_1) = \mathbb{R}$-rank(G_2).

Approximate Lattices

Approximate subgroup: An approximate subgroup of a group G is a subset $\Lambda \subset G$ such that Λ is symmetric, contains the identity element $e \in G$, and there exists a finite subset $F \subset G$ satisfying $\Lambda^2 \subset F\Lambda$.
A subset $P \subset G$ is called *relatively dense* if there exists a compact subset $K \subset G$ such that $G = KP$.
Uniform approximate lattice: An approximate subgroup Λ of a group G is called a *uniform approximate lattice* if it is uniformly discrete and relatively dense.

Approximate lattice: A discrete approximate subgroup Λ of a group G is called an *approximate lattice* if there exists a subset $F \subset G$ with finite Haar-measure such that $G = F\Lambda$.

Examples: (1) $\Lambda := \{a + b\sqrt{2} : a, b \in \mathbb{Z} \ \& \ |a - b\sqrt{2}| \leq 1\}$ is an approximate lattice in $\mathbb{R}$. It is coming from a higher rank lattice $\Gamma := \{(a + b\sqrt{2}, a - b\sqrt{2}) : a, b \in \mathbb{Z}\}$ in $\mathbb{R}^2$ via cut-and-project scheme.
(2) Matrices of the form $I + M$, where entries of M are taken from Λ and which satisfy the determinant-one condition, provide examples of approximate lattices in $\mathrm{SL}_n(\mathbb{R})$.

Strong Approximate Lattices and Cartan subgroups

Given a closed subset $Q \subset G$, its G-orbit in $\mathcal{C}(G)$ is the set $G \cdot Q := \{gQ \mid g \in G\}$. The invariant hull Ω_Q is defined as the closure of this orbit in $\mathcal{C}(G)$ with respect to the Chabauty–Fell topology.

Strong approximate lattice: A discrete approximate subgroup Λ of G is called *strong approximate lattice* if there exists a G-invariant Borel probability measure ν on Ω_Λ such that $\nu(\{\varnothing\}) = 0$.

The theory of approximate lattices and approximate groups has been extensively studied in recent years, with significant contributions from researchers including Terence Tao, Emmanuel Breuillard, Ben Green, Björklund, Hartnick, Hrushovski, Machado, and many others.

Theorem (Mandal and Singh, 2026): Let G be a linear semisimple Lie group without a compact factor and let $\Lambda = P_0(\Gamma, W_0)$ be a regular model set over a cut-and-project scheme (G, H, Γ), where H is a linear semisimple Lie group without a compact factor and Γ is an irreducible lattice in $G \times H$. Let C be a non-compact Cartan of G. If Γ is uniform, then $\Lambda^2 \cap gCg^{-1}$ contains a regular element and is an uniform approximate lattice in gCg^{-1}.

Thank you