Diagonal product of permutational wreath products

Estimates of the isoperimetry of a construction of groups of intermediate growth

Nathan Deloire [1]

[1] Laboratoire de Probabilités, Statistiques et Modélisation
Sorbonne Université

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Outline

Følner function: definition

Definition: Følner function

We call Følner function and we write $\mathrm{Føl}$ the following mapping:

$$\mathrm{Føl} : \begin{array}{rcl} \mathbb{R}_+ & \to & \mathbb{N} \\ n & \mapsto & \inf_{\Omega \subset C, \frac{\mathrm{Card}\ \Omega}{\mathrm{Card}\ \partial_{\mathrm{edge}}\Omega} \geqslant n} \mathrm{Card}\ \Omega \end{array}$$

Be careful: the domain of the function $\mathrm{Føl}$ is not always $\mathbb{R}_+$.

Known properties

Theorem: Existence of groups with prescribed isoperimetry [BZ21]

There exists a universal constant $C \in [1, +\infty)$ such that for any function $g : [1, +\infty) \to [1, +\infty)$ regular enough, there exists a group Δ such that:

$$\forall n \in \mathbb{N}\backslash\{0\}, g(n/C) \leqslant \mathrm{F}\o\mathrm{l}_{\Delta}(n) \leqslant g(Cn)$$

Property: Permutational wreath products of Grigorchuk groups [BE14]

Let Γ be a finite group, let G be a torsion Grigorchuk group, let X be an orbit of the action of G over the infinite binary rooted tree. Then the permutational wreath product $\Gamma \wr_X G$ has intermediate growth.

Question

Is it possible to construct groups with at least exponential prescribed isoperimetry and intermediate growth?

Outline

Grigorchuk groups: definition

See the blackboard.

The Grigorchuk groups: properties

Notation : Ω and Ω_0

We write $\Omega = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}^{\mathbb{N}}$.
We denote by Ω_0 the set of $\omega = (\omega_i)_{i \in \mathbb{N}} \in \Omega$ such that all three elements $\mathbf{0}, \mathbf{1}, \mathbf{2}$ appear an infinite number of times.

Property: intermediate growth of Grigorchuk groups [Gri84]

Let $\omega \in \Omega_0$. Then the Grigorchuk group G_ω is a torsion group of intermediate growth.

The Grigorchuk group orbits

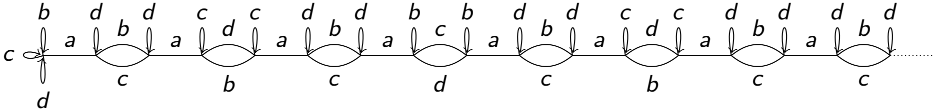

Figure: The connected component of the Schreier graph of the first Grigorchuk group $G_{(012)^\infty}$. Picture taken from [BE12]

Inverted orbit growth

Notation: $\delta(n)$, the growth of the inverted orbit

Given a Grigorchuk group G_ω acting on one of its orbits X, denote by δ the growth function of its inverted orbit, that is:

$$\delta : \begin{array}{rcl} \mathbb{N} & \to & \mathbb{N} \\ n & \mapsto & \sup_{w \in \{\mathfrak{a}, \mathfrak{b}_\omega, \mathfrak{c}_\omega, \mathfrak{d}_\omega\}^n} \mathrm{Card}\left\{\rho, \rho w_1, \rho w_2 w_1, \ldots, \rho w_n w_{n-1} \ldots w_2 w_1\right\} \end{array}.$$

Proposition: sublinearity of δ [BE14]

If $\omega \in \Omega_0$, then the function δ is sublinear.

Growth of the permutational wreath product

Property: Permutational wreath products of intermediate growth [BE14]

Let Γ be a finite group, let $\omega \in \Omega_0$, let X be an orbit of the action of G_ω over the infinite binary rooted tree. Then the permutational wreath product $\Gamma \wr_X G_\omega$ has intermediate growth.

Finite generating sets of a wreath product

Choice of the generating set of a wreath product

For all m, we endow the wreath product $\Delta_m = \Gamma_m \wr \mathbb{Z}$ with the finite generating set

$$S_{\Delta_m} = \{(\mathrm{id}_{\Gamma_m}, 1)\} \cup \{(a\delta_0, 0)\, , a \in A\} \cup \{(b\delta_{k_m}, 0)\, , b \in B\}$$

Careful: here, we're only interested in groups of the form $\Gamma_m \wr \mathbb{Z}$.

Diagonal product of wreath products

Diagonal product of wreath products: definition

Let be an increasing sequence $(k_m)_{m\in\mathbb{N}}$, diverging towards $+\infty$. We then call Brieussel-Zheng diagonal product the subgroup Δ of $\left(\prod_{m\in\mathbb{N}}\Gamma_m\right)\wr\mathbb{Z}$ generated by the generating set

$$S_\Delta=\{((\mathrm{id}_{\Gamma_m})_{m\in\mathbb{N}},1)\}\cup\{((a\delta_0)_{m\in\mathbb{N}},0)\,,a\in A\}\cup\{((b\delta_{k_m})_{m\in\mathbb{N}},0)\,,b\in B\}$$

where δ_i is the sequence whose value is 1 at the position i and 0 everywhere else.
We endow the diagonal product Δ with the generating set S_Δ above.

Remark: local geometry of the diagonal product

$\forall n\in\mathbb{N},\mathcal{B}_\Delta(0,n)\simeq\mathcal{B}_{\Delta_{\leqslant\kappa(n)+1}}(0,n)$.

Outline

Finite generating sets of permutational wreath products

Choice of the generating set of a permutational wreath product

Recall that X is the orbit of 1^∞ by the action of G_ω. Let $(k_m)_{m\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence of elements in the orbit of 1^∞ at distance $|k_m|$ of 1^∞, which diverges towards $+\infty$.

For all m, we endow the permutational wreath product $\Delta_m = \Gamma_m \wr G_\omega$ with the following finite generating set:

$$\begin{aligned} S_{\Delta_m} = & \quad \left\{ \left((e_{\Gamma_m})_{x\in X}, \mathfrak{a} \right), \left((e_{\Gamma_m})_{x\in X}, \mathfrak{b} \right), \left((e_{\Gamma_m})_{x\in X}, \mathfrak{c} \right), \left((e_{\Gamma_m})_{x\in X}, \mathfrak{d} \right) \right\} \\ & \cup \quad \left\{ \left((\delta_{1^\infty, x}(a_m))_{x\in X}, e_{G_\omega} \right), a_m \in A_m \right\} \\ & \cup \quad \left\{ \left((\delta_{k_m, x}(b_m))_{x\in X}, e_{G_\omega} \right), b_m \in B_m \right\}. \end{aligned}$$

Diagonal product of permutational wreath products

Diagonal product of permutational wreath products

Let $(k_m)_{m\in\mathbb{N}} \in X^{\mathbb{N}}$ be such that $(|k_m|)_{m\in\mathbb{N}}$ is an increasing sequence, diverging towards $+\infty$. We then call Brieussel-Zheng diagonal product the subgroup Δ of $\left(\prod_{m\in\mathbb{N}} \Gamma_m\right) \wr G_\omega$ generated by the following generating set:

$$\begin{aligned} S_\Delta = & \quad \left\{ \left((e_{\Gamma_m})_{x\in X, m\in\mathbb{N}}, \mathfrak{a} \right), \left((e_{\Gamma_m})_{x\in X, m\in\mathbb{N}}, \mathfrak{b} \right) \right\} \\ & \cup \quad \left\{ \left((e_{\Gamma_m})_{x\in X, m\in\mathbb{N}}, \mathfrak{c} \right), \left((e_{\Gamma_m})_{x\in X, m\in\mathbb{N}}, \mathfrak{d} \right) \right\} \\ & \cup \quad \left\{ \left((\delta_{1^\infty, x}(a_m))_{x\in X, m\in\mathbb{N}}, e_{G_\omega} \right), a \in A \right\} \\ & \cup \quad \left\{ \left((\delta_{k_m, x}(b_m))_{x\in X, m\in\mathbb{N}}, e_{G_\omega} \right), b \in B \right\}. \end{aligned}$$

Bounding the Følner function of Δ

For the following proposition, we write:

- The finite groups Γ_m are generated by their subgroups A and B, and $\ell(m) = \log|\Gamma_m|$;
- The k_m, used in the definition of the generating sets S_{Δ_m} of Δ_m, indicate "where the lamps are lit on by the group B", and the function $\kappa(n) = \max\{m \in \mathbb{N}, |k_m| \leqslant n\}$ is the reciprocal of the sequence $(|k_m|)_{m\in\mathbb{N}}$;
- $D(n) = \mathrm{diam}\, V_n$ is the diameter of an n-Følner set of the group G_ω;
- $\delta(n)$ is the inverted orbit growth function of the group G_ω
- We assume $\forall m \in \mathbb{N}, k_{m+1} \geqslant 2k_m$ and $\forall m \in \mathbb{N}, \ell(m+1) \geqslant 2\ell(m)$.

Main result: Bounding the function $\mathrm{Føl}_\Delta$ [D., 26$^+$]

We have, asymptotically:

$$\mathrm{e}^{n\ell\circ\delta\circ\kappa(n)} \lesssim \mathrm{Føl}_\Delta(n) \lesssim \mathrm{e}^{D(n)\ell\circ\kappa(D(n))}$$

Remarks

Main result: Bounding the function $\mathrm{F\o l}_\Delta$ [D., 26^+]

We have, asymptotically:

$$\mathrm{e}^{n\ell\circ\delta\circ\kappa(n)} \lesssim \mathrm{F\o l}_\Delta(n) \lesssim \mathrm{e}^{D(n)\ell\circ\kappa(D(n))}$$

Remark on the variables

- The functions κ and ℓ are parameters, that may be chosen (as long as $\forall m \in \mathbb{N}, \ell(m+1) \geqslant \ell(m)$ and $\forall m \in \mathbb{N}, k_{m+1} \geqslant k_m$).
- The function $D(n)$ only depends upon ω, is superlinear, and is bounded above by the Følner function.
- The function δ only depends upon ω, is sublinear, and, for any function $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that $\forall R \in \mathbb{R}_+^*, g(2R) \leqslant 2g(R) \leqslant g(2,46R)$, there exists ω such that $\delta \simeq g$.

Examples

Main result: Bounding the function $\mathrm{F}\text{ø}\mathrm{l}_\Delta$ [D., 26+]

We have, asymptotically:

$$\mathrm{e}^{n\ell\circ\delta\circ\kappa(n)} \lesssim \mathrm{F}\text{ø}\mathrm{l}_\Delta(n) \lesssim \mathrm{e}^{D(n)\ell\circ\kappa(D(n))}$$

Example

Let $C \in \mathbb{R}_+$ such that $C > 1$. Choosing $\omega = (012)^\infty$, $\kappa(n) = \log n$ and $\ell(m) = \mathrm{e}^{(C-1)n^{\frac{1}{\alpha}}}$ with $\alpha \simeq 0,767...$. We see that there exists a group Δ such that:

$$\exp\left(n^C\right) \lesssim \mathrm{F}\text{ø}\mathrm{l}_\Delta(n) \lesssim \exp\left(\mathrm{e}^{C'\log(n)^{1/\alpha}}\right)$$

where $C' = \frac{C-1}{(1-\alpha)^{1/\alpha}}$. Therefore, since $\frac{1}{\alpha} < 2$, we have:

$$\exp\left(n^C\right) \lesssim \mathrm{F}\text{ø}\mathrm{l}_\Delta(n) \lesssim \exp\left(n^{\log n}\right).$$

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