Do the symmetric groups live inside L_1?

Cosmas Kravaris
graduate student, Princeton.

July 8, 2026

Basic Fact: Different word metrics on a f.g. group have the same bi-Lipschitz geometry: For each $< S >=< S' >= G$ there exists a **rescaling** $r > 0$ and **distortion** $D > 1$ such that

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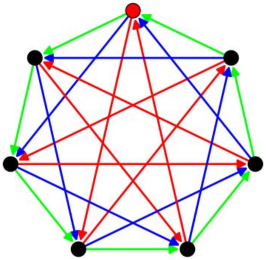
—New way the story breaks: embedability into L_1.
(Drawing an analogy to the lamplighter group is a crucial step.)
—Related problem where the lamplighter analogy helps; joint work with Eduardo Silva.

What is the analogous setting in finite groups?



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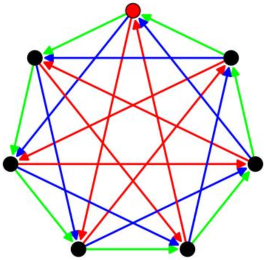

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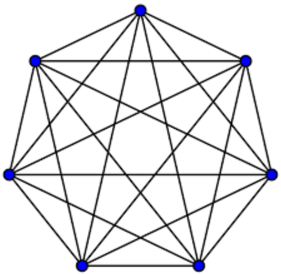

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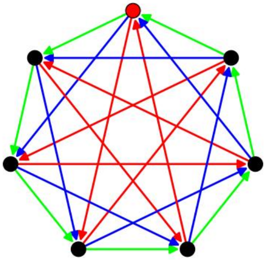

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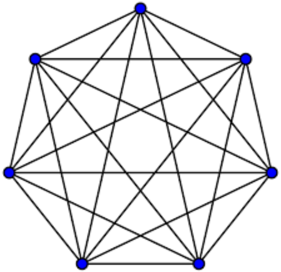

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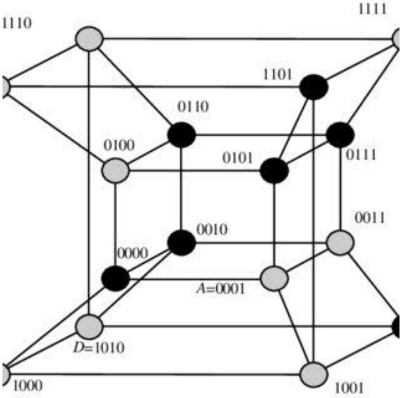

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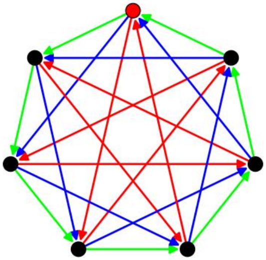

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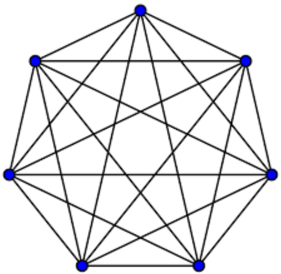

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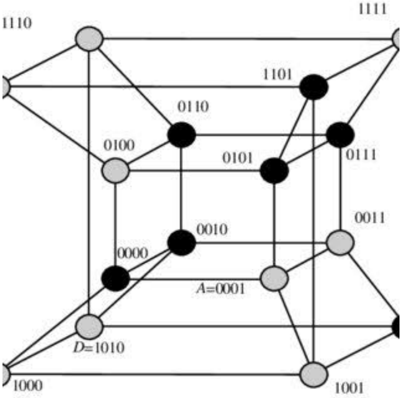

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$$We\ \ want\ \ bounded\ \ geometry: \ \sup_n |S_n| < \infty \ \ and \ \ |G_n| \to \infty.$$

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Note: the diameter is a bi-Lipschitz invariant of a metric space.

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UPSHOT same group – different bounded size generating sets – different geometry!

Question of Lubotzky-Weiss: Is expansion a group property?

Recall: a sequence of graphs $((V_n, E_n))_n$ is a **family of expander graphs** if there exists $\kappa > 0$ such that for any n, $S \subset V_n$ of size $|S_n| < |V_n|/2$, we have

$$|\partial S| = \{(u,v) \in E : u \in S,\ and\ v \notin S\} \geq \kappa |S|$$

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If $(Cay(G_n, S_n))_n$ are expander graphs with $|S_n| = O(1)$,
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Result: This contrast also holds for bi-Lipschitz embedability into L_1.

What is a metric embedding into L_1?

Definition: given a metric space (M, d), a map $f : M \to L_1([0, 1])$ has **bi-Lipschitz distortion** $D > 1$ when

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3—Embedability into L_1 is equivalent into a measure space of **cuts (aka walls)** for which the distance $d(x,y)$ is estimated by the measure of the cuts which separate x from y.
(e.g. a group action on a $CAT(0)$-cube complex gives such an action.)

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Theorem (Kassabov, 2007)

The symmetric groups admit Cayley graphs which are bounded degree expanders.

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Bounded degree expanders cannot embed into L_1 *with uniformly bounded distortion.*

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In sharp contrast, we have:

Theorem (K. 2026+)

$Cay(Sym_n, \{(12), (123\ldots n)\})$ *embeds into L_1 with biLipschitz distortion* < 1000.

Drawing an analogy with the lamplighter/lampshuffler groups.

$Cay(Sym_n, \{(12), (123\dots n)\})$ resembles a **lamplighter group** $\mathbb{Z}/n \wr \mathbb{Z}/n$:
–$c = (123\dots n)$ resembles a pointer movement.
–$t := (12)$ resembles a change in the "clock/counter" at the pointer position.
Evidence for L_1-embedability: Naor-Peres(2008) showed the standard word metric on $\mathbb{Z}/n \wr \mathbb{Z}/n$ embeds into L_1 with biLipschitz distortion $O(1)$.

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The presence of the minimum is like a *a lamplighter who forgot their position.*
For small distances the metric resembles a lamplighter/lampshuffler.
For large distances the metric resembles a median-type quantity.
Difficulty: an embedding into L_1 must preserve both small and large distances!

We always have $diam_{\mathbb{Z}/n}(\{0,l\} \cup \{p : \pi^{-1}(p) \neq \tau^{-1}(p-l)\}) < n/2$ and when $\min_{l\in\mathbb{Z}/n} \sum_{k\in\mathbb{Z}/n} d_{\mathbb{Z}/n}(\pi(k)-l,\tau(k)) < n/100$ then the optimizer l is the one that fixes 99% of the distances $d_{\mathbb{Z}/n}(\pi(k)-l,\tau(k)) = 0$. Thus:

$$d(\tau,\pi) \asymp \min_{l\in\mathbb{Z}/n}\left(\sum_{k\in\mathbb{Z}/n} d_{\mathbb{Z}/n}(\pi(k)-l,\tau(k)) + diam_{\mathbb{Z}/n}(\{0,l\} \cup \{p : \pi^{-1}(p) \neq \tau^{-1}(p-l)\})\right)$$

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$$T_1 = \min_{l\in\mathbb{Z}/n} \sum_{k\in\mathbb{Z}/n} d_{\mathbb{Z}/n}(\pi(k)-l,\tau(k)) = \min_{l\in\mathbb{Z}/n} \sum_{k\in\mathbb{Z}/n} d_{\mathbb{Z}/n}(\pi(k)-\tau(k),l)$$

so in the term T_1, the point $l \in \mathbb{Z}/n$ is the **median** of the set $\{\pi(k)-\tau(k) | k \in \mathbb{Z}/n\}$.

$$\asymp \frac{1}{n} \sum_{k\in\mathbb{Z}/n} \sum_{k'\in\mathbb{Z}/n} d_{\mathbb{Z}/n}(\pi(k)-\tau(k), \pi(k')-\tau(k')) \asymp \sum_{k} \sum_{k'} |e^{2\pi i(\pi(k)-\pi(k'))/n} - e^{2\pi i(\tau(k)-\tau(k'))/n}|$$

hence we have an embedding for T_1: $\pi \mapsto (e^{2\pi i(\pi(k)-\pi(k'))/n})_{k,k'\in\mathbb{Z}/n}$.
For the term T_2 we modify the embedding $\mathbb{Z}/2 \wr \mathbb{Z}/n \to L_1$ of (Naor–Peres 2008).

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–The goal of the puzzle is to sort the tiles in lexicographical order (i.e. we walk on the Cayley graph and return to the identity).

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–The tiles get shuffled by moving one of the 4 neighboring tiles into the vacant spot. This gives us the Cayley graph of a finite group.

–The goal of the puzzle is to sort the tiles in lexicographical order (i.e. we walk on the Cayley graph and return to the identity).

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K.–Silva (2026+): Drift/Speed: if I shuffle the 15-puzzle on the $M \times M$-torus for t steps, then how many steps do I need to solve it?

$$\mathbb{E}[\#steps] \asymp \frac{t}{\sqrt{\log t}} \quad when \quad 1 \leq t \lesssim M^2 \log M.$$

$$\mathbb{E}[\#steps] \asymp M\sqrt{t} \quad when \quad M^2 \log M \lesssim t \lesssim M^4.$$

$$\mathbb{E}[\#steps] \asymp M^3 \quad when \quad t \gtrsim M^4.$$

Thank you for listening!

Questions?