Squares of conjugacy classes of S_n

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Joint work with:
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The Goal

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Our Result *(Keller, Lifshitz, Sheinfeld – Forum Math. Sigma '24)*

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The Standard Tool: Convolution

To prove that A^2 hits a target conjugacy class C, we use the convolution operator $M_C : L^2(S_n) \to L^2(S_n)$:

$$M_C f := f * 1_C := \sum_x f(x) 1_C(x^{-1}g)$$

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If we can show that this inner product is greater than 0, we win.

The Method of Larsen and Shalev

The Algebraic Dream: Since $1_C, 1_A$ are class functions, representation theory gives us

$$\langle 1_A, 1_A * 1_C \rangle = \sum_{\chi} \frac{\chi(C)\chi(A)^2}{\chi(1)} = 1 + \sum_{\chi \neq 1} \frac{\chi(C)\chi(A)^2}{\chi(1)}$$

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The $1/4$ Wall

The character bounds depend on the number of small cycles of A, C. They could have many small cycles so the method got stuck at $n^{1/4}$ cycles.

If we manage to reduce the problem to conjugacy classes with less small cycles, we could improve the bound.

The Pivot: Removing cycles

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Look at the inner product now:

$$\langle 1_A, M_C(1_A)\rangle = \sum_{\lambda \in \textbf{Eigenvalue of } \mathbf{M_C}} \lambda \times \underbrace{\left(\text{Norm of } 1_A \text{ restricted to } V_\lambda\right)}_{\textbf{???}}$$

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compared to

$$\langle 1_A, M_C(1_A) \rangle = \sum_{\chi} \frac{\chi(C)}{\chi(1)} \cdot \chi(A)^2$$

The Challenge: Since 1_A is no longer a class function, we don't know to bound the norm of 1_A restricted to V_λ.

The Secret Weapon: Hypercontractivity

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Representation theory + Hypercontractivity = Better Bounds

Work in Progress

In Preparation (Keller, Lifshitz, Marmor, Sheinfeld):

If $|A| \geq |S_n| \cdot (n!)^{-\epsilon}$ then $A_n \subseteq A^2$

Previous Result *(Keller, Lifshitz, Sheinfeld – Forum Math. Sigma '24)*

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Thank You!