Lower semi-continuity of the fundamental group

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GONG SHOW lightning talk

- LMS-HIMR Research school on growth and expansion in groups
- University of Warwick, United Kingdom
- July 6th – 10th, 2026

Topology

Gromov-Hausdorff convergence	=	convergence of metric spaces

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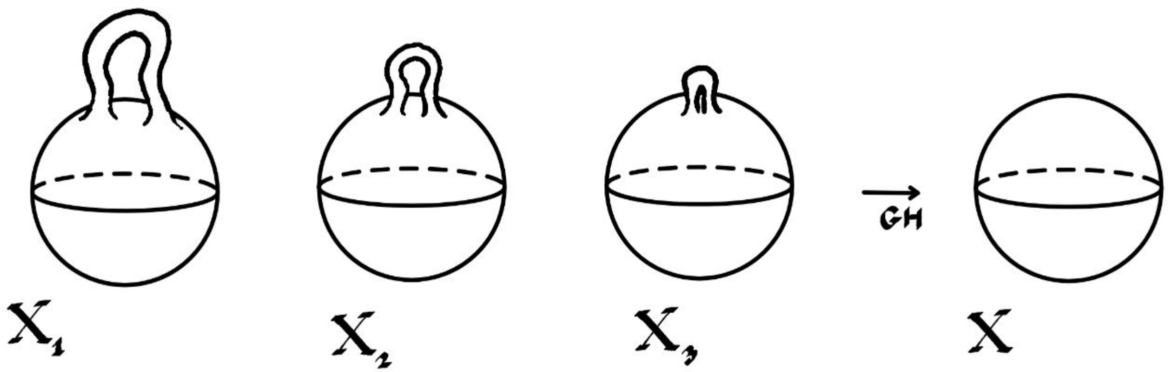
e.g.



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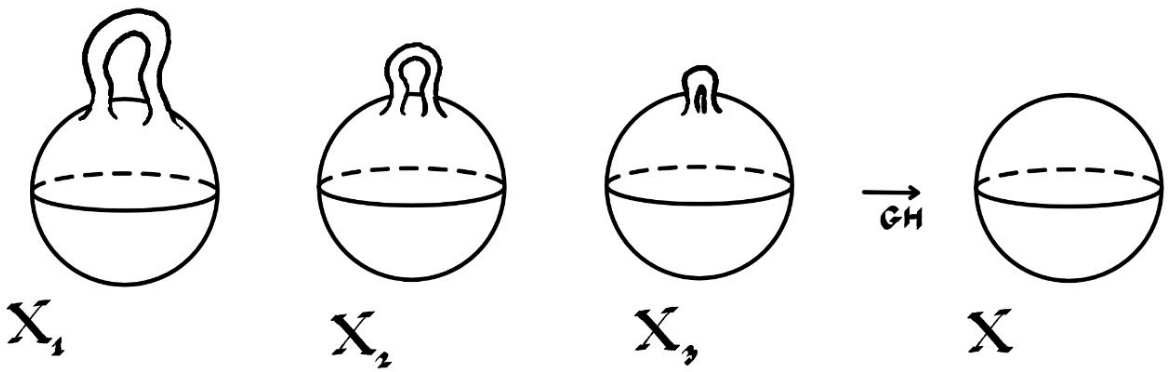

Question: How to relate the topology of the limit to that of the sequence?

Limitations

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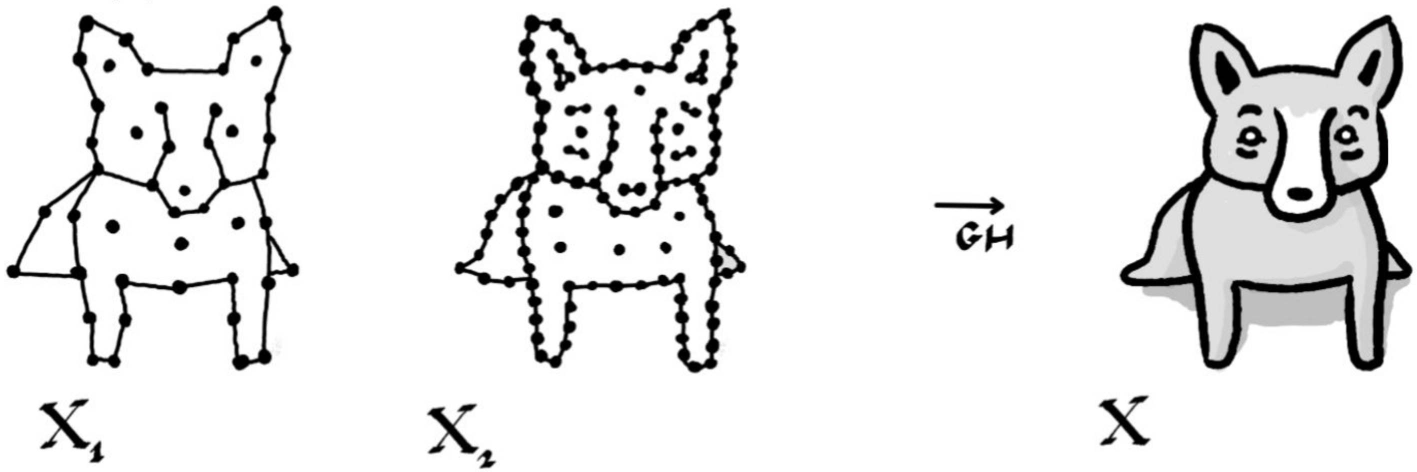
- Topology can appear !



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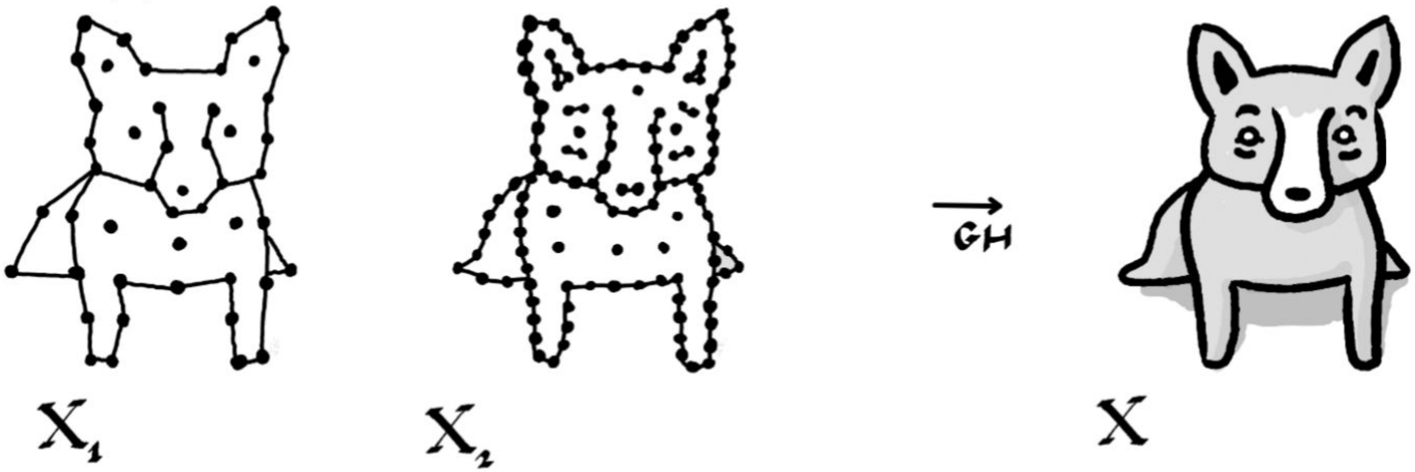

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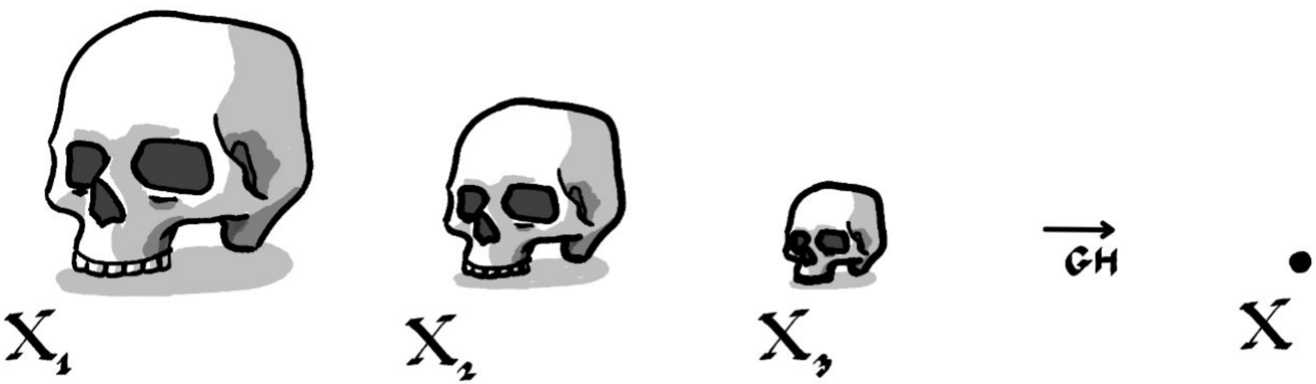

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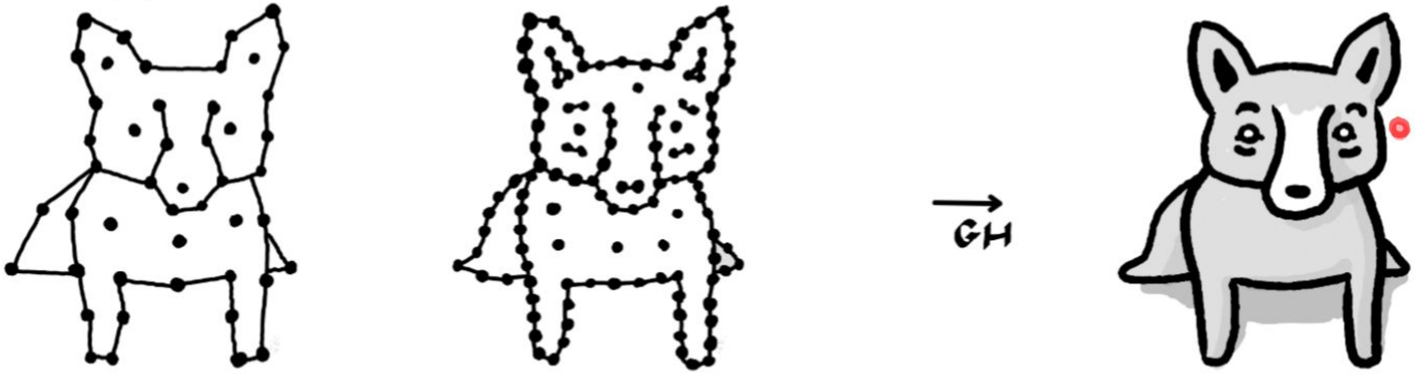

X_1 X_2 $\xrightarrow{GH}$ X

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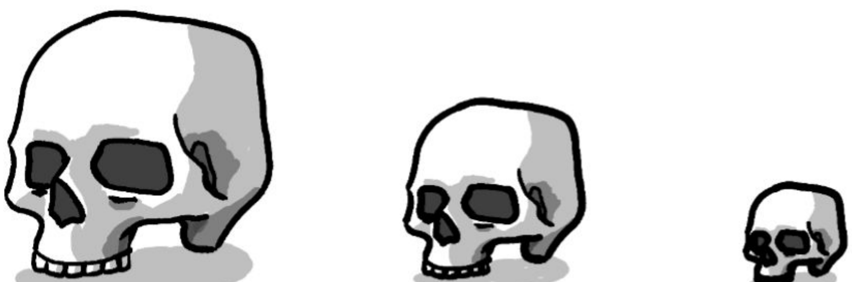


X_1 X_2 X_3 $\xrightarrow{GH}$ $\bullet$ X

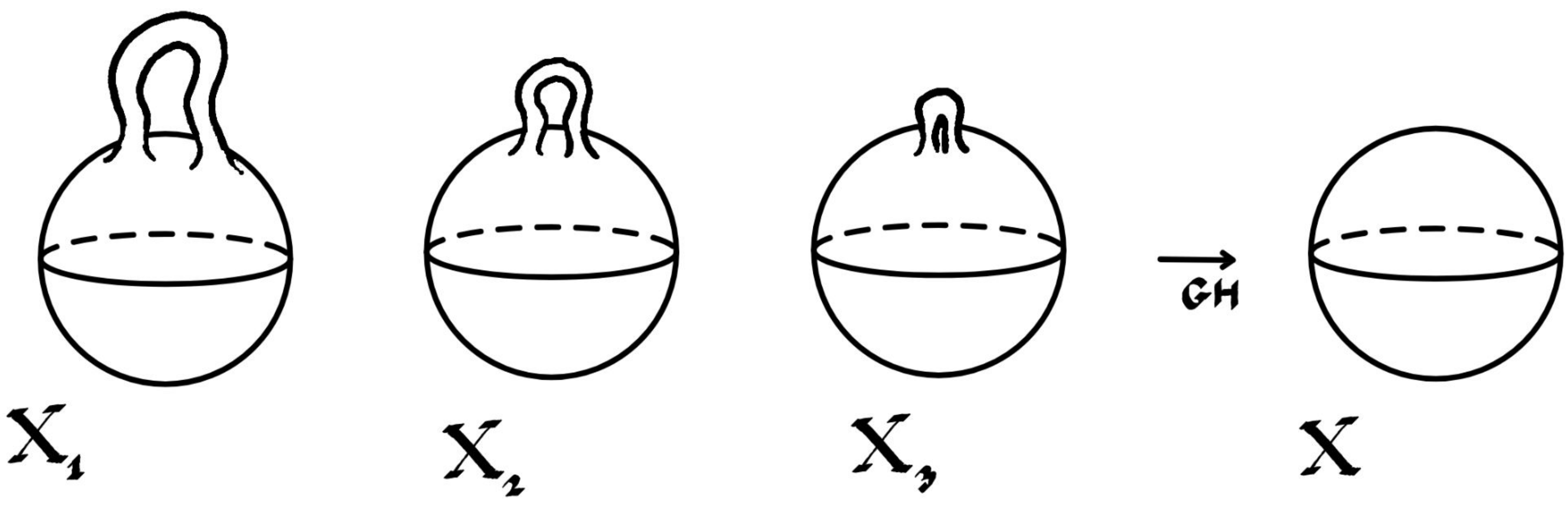
• Compact Setting (circa 1985)

Theorem (Lower semicontinuity of π_1, Rakotoniaina)

Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of compact, geodesic spaces, converging to a compact, geodesic, locally-contractible space, X.

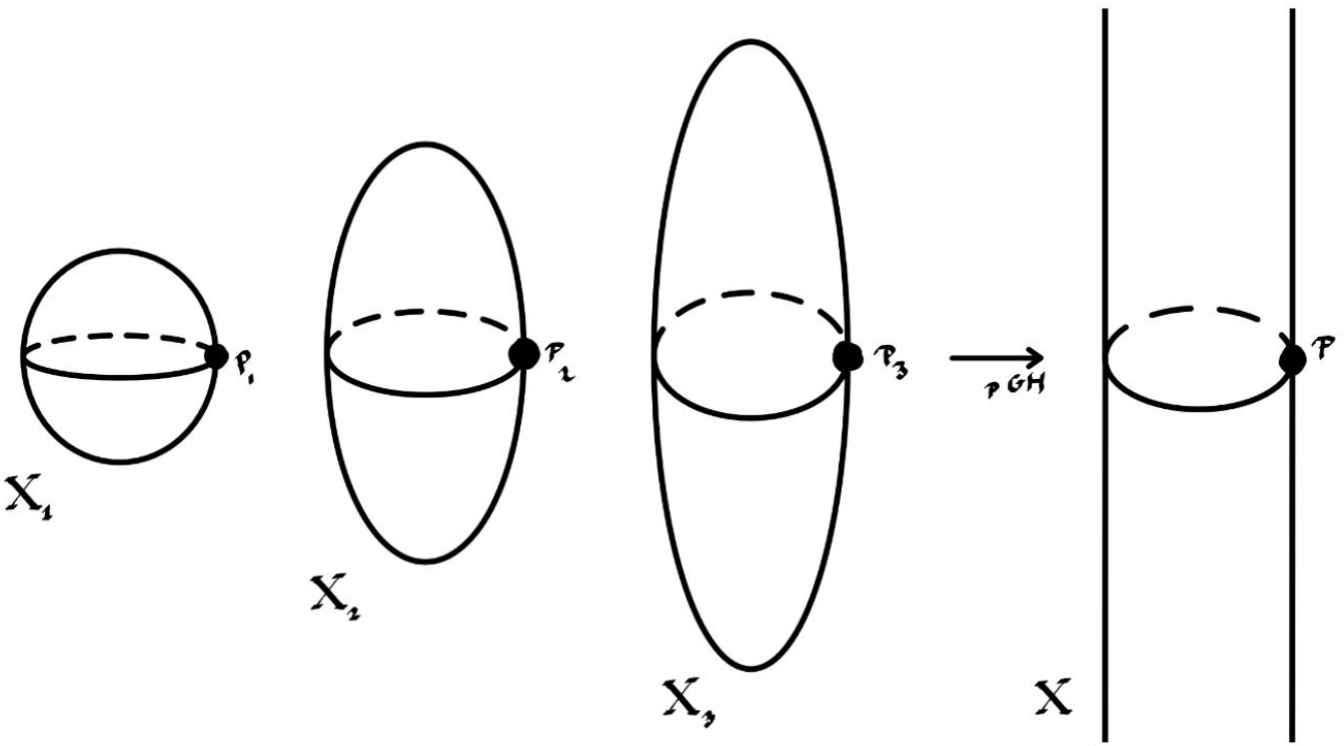
Then, for large enough n, there are surjective group homomorphisms

$$\pi_1(X_n) \twoheadrightarrow \pi_1(X).$$



Corollary: Compact, locally-contractible limits of compact, geodesic simply-connected spaces are simply-connected.

non-compact lower semi-continuity of π_1?



$\pi_1(X_n) \cong \{0\} \not\twoheadrightarrow \mathbb{Z} \cong \pi_1(X_n)$

Requiring symmetry?

- A metric space, X, is homogeneous if $\mathrm{Iso}(X)$ acts transitively.

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CONJECTURE 0: Suppose that $\{(X_n, p_n)\}_{n=1}^{\infty}$ a sequence of proper, pointed, geodesic spaces, converging to a proper, pointed, geodesic, space, (X, p).

Then, if X_n are $\begin{cases} \text{simply-connected and} \\ \text{homogeneous} \end{cases}$,

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FALSE! There are homogeneous metrics on $\mathbb{S}^3$ (rescaled Berger metrics) converging to $\mathbb{R}^2 \times \mathbb{S}^1$.

Add Discrete Symmetry

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Then, if X_n are $\begin{cases} \text{simply-connected and} \\ \exists G_n \leq_{disc} \mathrm{Iso}(X_n) : \mathrm{diam}(X_n/G_n) \searrow 0 \end{cases}$,

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- Known when X is locally-contractible.
- Interesting case: X an ∞-dim.l group.

Outline

A. X is isometric to a nilpotent group, G.

- Theory of approximate groups [BGT].

$\xRightarrow{\text{WLOG}}$ $\{G_n \curvearrowright X_n\}$ are "almost nilpotent".

$\Longrightarrow$ $\lim G_n = G \curvearrowright X$ nilpotent, acting trans.

$\Longrightarrow$ $X \cong G$.

B. X is locally-contractable iff G is Lie.

- Theorem of Gleason-Yamabe (Hilbert 5).

$\Longrightarrow$ $G/_{\text{small compact}}$ is Lie.

THANKS!

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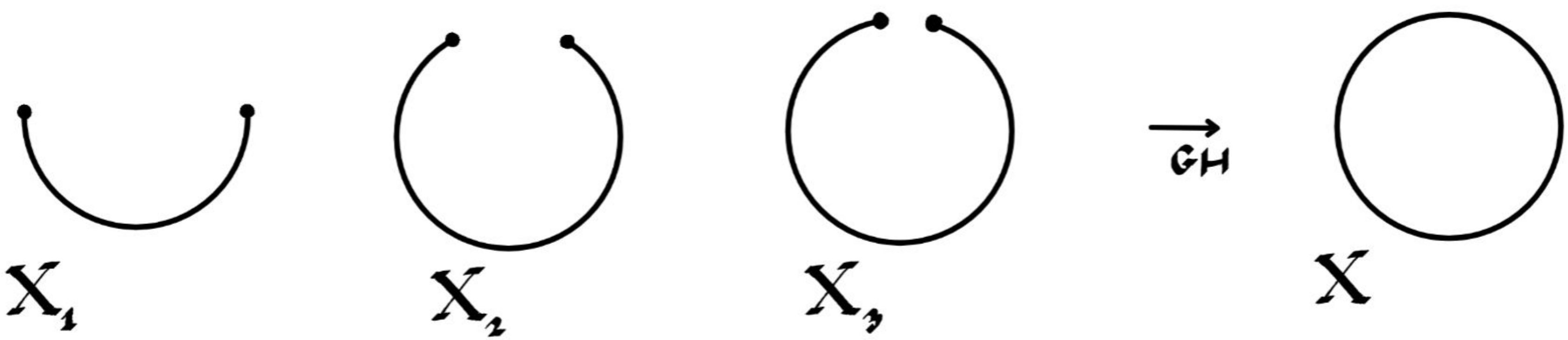


FIGURE 4: A sequence of semi-circles, converging to a circle.

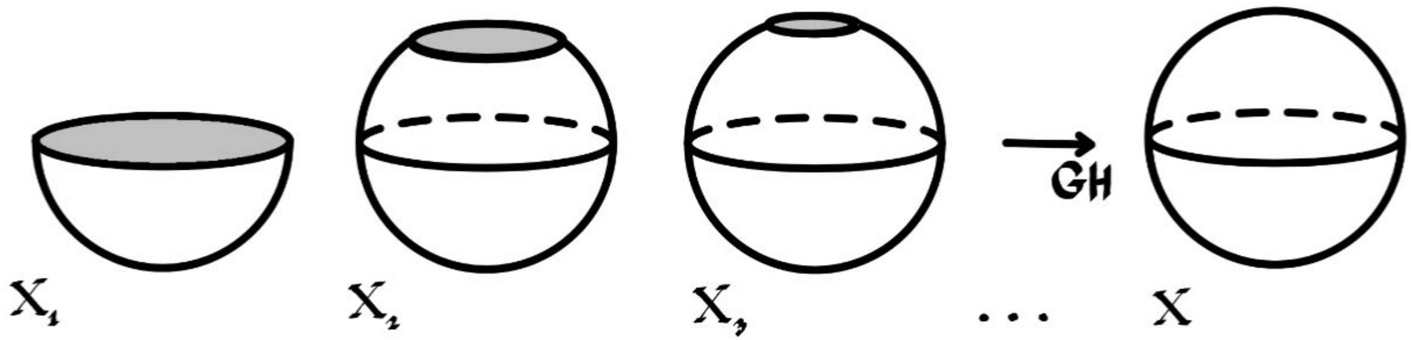


Figure 2: A sequence of spheres with smaller and smaller caps removed, converging to a sphere.

Example 0.0.0 (Griffiths twin cone) It might at first appear that it is not possible for a sequence of compact, geodesic, *simply connected* metric spaces to converge in the global Gromov-Hausdorff sense to a space which isn't even semi-locally-simply-connected. The Griffiths Twin Cone (defined as the twin cone over the earring space), however, is a non-semi-locally-simply-connected space, which is the limit of contractible (and hence simply-connected) spaces (see [BRA14], [BRA23], and [GRI54]).

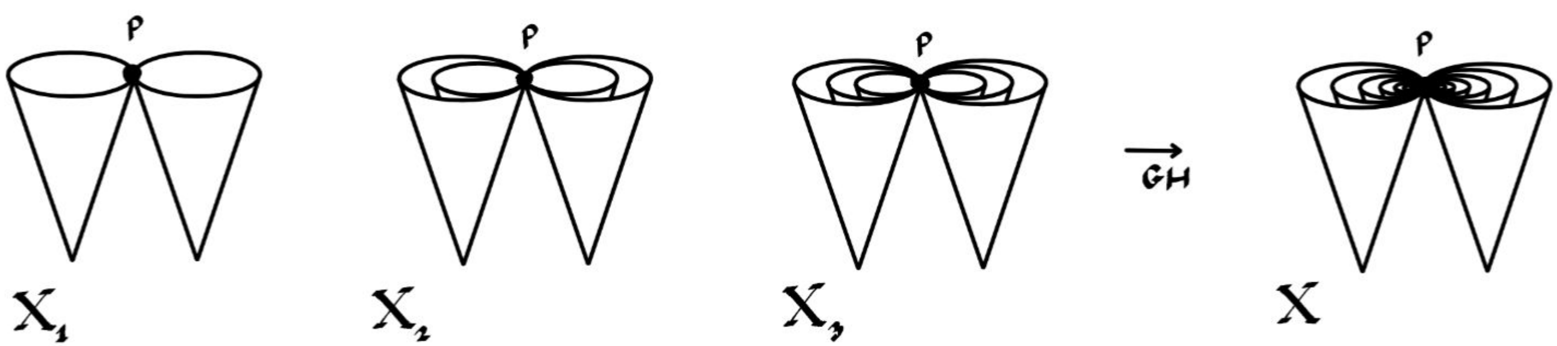
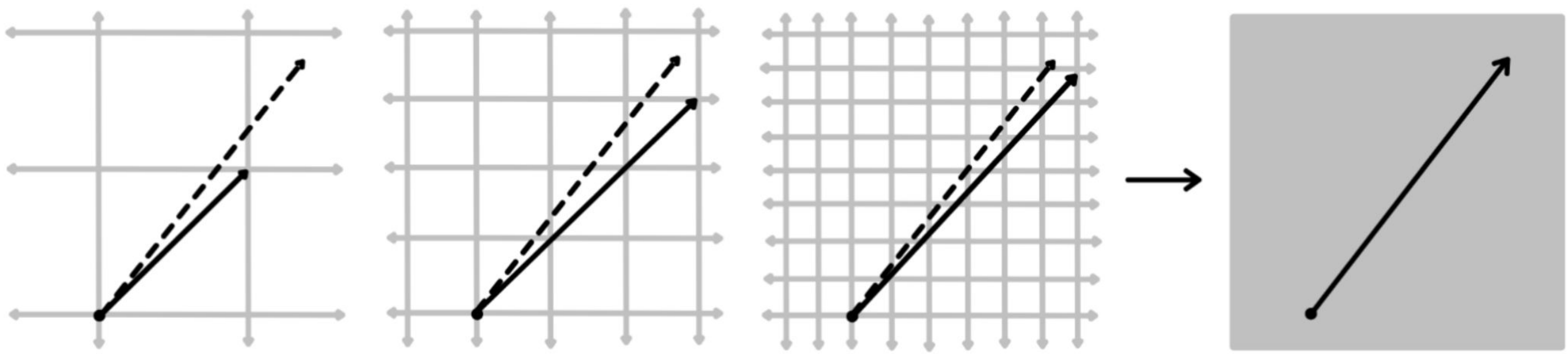
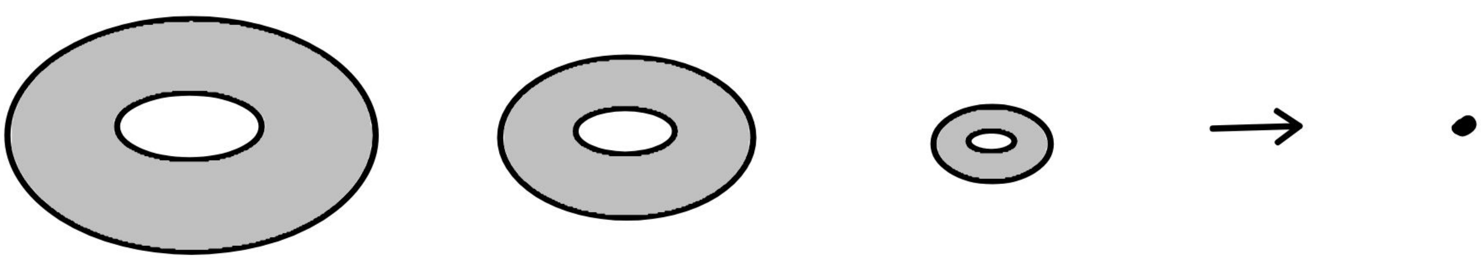


FIGURE 1: *A sequence of partial double-cones, converging to the Griffiths twin cone.*

DEFINITION: A sequence of proper, geodesic spaces is said to consist of almost-homogeneous spaces if, for each n, $\exists \Gamma_n \leq \mathrm{Iso}(X_n)$ discrete, with $\mathrm{diam}(X_n / \Gamma_n) \to 0$.



$\mathbb{Z}^2 \circlearrowright \mathbb{R}^2$ $\quad$ $(\tfrac{1}{2}\mathbb{Z})^2 \circlearrowright \mathbb{R}^2$ $\quad$ $(\tfrac{1}{3}\mathbb{Z})^2 \circlearrowright \mathbb{R}^2$ $\quad$ $\mathbb{R}^2 \circlearrowright \mathbb{R}^2$



Folktale

Perelman's finiteness theorem:

Fix $N \in \mathbb{N}$, $\kappa \in \mathbb{R}$, $v, D > 0$. Up to homeomorphism, there are only *finitely-many* Riemannian manifolds satisfying:

$$\boxed{\dim = N,\ \sec > \kappa,\ \operatorname{diam} < D,\ \operatorname{vol} > v.}$$

- By contradiction, a sequence $X_1, X_2, \ldots, X_n, \ldots$.
- Gromov's theory of convergence of metric spaces; precompactness.
- Topological stability.

G. Perelman

THEOREM: (Zamora, 2024) Suppose that $(X_n, p_n) \xrightarrow[PGH]{} (X, p)$ consists of almost-homogeneous spaces, X: SLSC.

Then, for large enough n, $\exists H_n \geq \pi_1(X)$ and surjective group homomorphisms

$$\pi_1(X_n) \twoheadrightarrow H_n .$$

Example 1.9. Let Y be $\mathbb{S}^1$ with its standard metric of length 2π and Z_n be $\mathbb{S}^3$ with the round (bi-invariant) metric of constant curvature $1/n^2$. Let X_n be the quotient $(Y \times Z_n)/\mathbb{S}^1$ where $\mathbb{S}^1$ acts on $Y \times Z_n$ as follows:

$$z(w, q) = (wz^{-1}, zq) : z, w \in \mathbb{S}^1, q \in \mathbb{S}^3.$$

Then X_n is isometric to $\mathbb{S}^3$ equipped with a re-scaled Berger metric. The sequence X_n consists of simply connected homogeneous spaces, but its pointed Gromov–Hausdorff limit is $\mathbb{S}^1 \times \mathbb{R}^2$, which is not simply connected.

Example 1.10. Define a "dot product" $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^6$ in $\mathbb{R}^4$ as

$$(a \cdot b) := (a_1 b_2, a_1 b_3, a_1 b_4, a_2 b_3, a_2 b_4, a_3 b_4), \;\; a, b \in \mathbb{R}^4.$$

With it, define a group structure on $\mathbb{R}^4 \times \mathbb{R}^6$ as

$$(a, x) \cdot (b, y) := (a + b, x + y + (a \cdot b)), \;\; a, b \in \mathbb{R}^4, \;\; x, y \in \mathbb{R}^6.$$

Let G be the above group equipped with a left invariant Riemannian metric. For each n, define the subgroups $K \leq K_n \triangleleft G_n \leq G \cong \mathbb{R}^4 \times \mathbb{R}^6$ as:

$$\begin{aligned} K &:= \{0\} \times \mathbb{Z}^6 \\ K_n &:= (n\mathbb{Z}^4) \times \mathbb{Z}^6 \\ G_n &:= \left(\frac{1}{n}\mathbb{Z}^4\right) \times \left(\frac{1}{n^2}\mathbb{Z}^6\right). \end{aligned}$$

The sequence $X_n := G/K_n$ converges in the pointed Gromov–Hausdorff sense to $X := G/K$. Since the sequence of finite groups G_n/K_n acts on X_n with $\mathrm{diam}(X_n/(G_n/K_n)) \to 0$, the sequence X_n consists of almost homogeneous spaces. A direct computation shows that the abelianization of $\pi_1(X_n) = K_n$ is isomorphic to

$$\mathbb{Z}^4 \oplus \left(\mathbb{Z}/n^2\mathbb{Z}\right)^6.$$

Then it is easy to see that $\pi_1(X) = K \cong \mathbb{Z}^6$ is not a quotient of $\pi_1(X_n)$ for any n.

DEFINITION 1.2.2 Let X be a proper metric space, and distinguish a point $p \in X$. Then, define $d_p : \mathrm{Iso}(X) \times \mathrm{Iso}(X) \to [0, +\infty]$ by

$$d_p(g,h) := \inf_{r>0} \left\{ \frac{1}{r} + \sup_{x \in B_r(p)} \{d(g(x), h(x))\} \right\} .$$

The function, d_p, endows $\mathrm{Iso}(X)$ with many nice properties, as is outlined in the following theorem:

Proposition 1.F *Let X be a proper metric space, and distinguish a point $p \in X$. Then, the following hold:*

(i) $(\mathrm{Iso}(X), d_p)$ *is a metric space.*

(ii) The metric d_p is left-invariant with respect to composition; that is,

$$\forall f, g, h \in \mathrm{Iso}(X) : \quad d_p(g,h) = d_p(f \circ g, f \circ h) .$$

(iii) The space $(\mathrm{Iso}(X), d_p)$ *is proper.*

(iv) The metric d_p induces the compact-open topology on $\mathrm{Iso}(X)$ *(or, equivalently, the topology of uniform convergence on compact sets).*

(v) Closed subgroups generate closed orbits; that is, for each closed subgroup $\Gamma \leq \mathrm{Iso}(X)$, we have that, for each $x \in X$, the orbit $\Gamma \cdot x := \{y \in X \mid \exists f \in \Gamma : f(y) = x\}$ is a closed subset of X.

- There are other nice answers to our main question in more specific cases:

- X_n, d-dim, compact RM, $\mathrm{vol}(X_n) \geq \varepsilon > 0$, $X_n \to_{GH} X$.
    - Cheeger, Colding: X RM

      smooth limit

      $$\mathrm{Ric}(X_n) \geq k \implies X_n \cong_{\text{diffeo}} X \text{ eventually.}$$

    - Perelman:

      limit with singularities

      $$\sec(X_n) \geq k \implies X_n \cong_{\text{homeo}} X \text{ eventually.}$$

Theorem 2.J (Wang, 2022) *Let* $\{(M_i, g_i)\}_{i=1}^{\infty}$ *be a sequence of compact, Riemannian n-manifolds, with* $\mathrm{Ric}(M_i) \geq -(n-1)$ *for each* i *and* $\mathrm{diam}(M_i) \leq D$*, for some fixed* D*. Suppose that* $M_i \to_{GH} X$*. Then,* X *is semi-locally-simply-connected.*

Theorem 2.K (Ferry-Okun, 1995) *Let* M *be a closed smooth manifold of dimension greater or equal to 3. Suppose that* X *is a compact, connected Riemannian manifold. Then, the following are equivalent:*

(i) *There exists a sequence of Riemannian metrics,* $\{g_n\}_{n=1}^{\infty}$*, on* M *such that* $(M, g_n) \to_{GH} X$.

(ii) *There exists a continuous function* $f : M \to X$ *inducing a surjective group homomorphism* $f_* : \pi_1(M) \to \pi_1(X)$.

In particular, any simply-connected Riemannian manifold, X, is approximated in the global Gromov-Hausdorff sense arbitrarily-well by "distortions" of $M = \mathbb{S}^3$ via the Riemannian metrics, g_n, in Theorem 2.K.