

Diameter bounds for arbitrary finite groups

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COROLLARY: Conjecture 1 $\implies$ Conjecture 2

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(Dona 2019)

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