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Cayley graphs of quasirandom groups

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We are only interested in Cayley graphs.

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A finite group G is *ε-quasirandom* if every nontrivial irreducible complex representation of G has degree at least $|G|^{\varepsilon}$.

Typical examples are simple groups of Lie type of bounded rank.
Non-examples are $\mathrm{Sym}(n)$, $\mathrm{Alt}(n)$, $\mathrm{GL}_n(2)$.

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Theorem (Nikolov-Pyber '11)

G is ε-quasirandom if and only if acts nontrivially only on sets of cardinality at least $|G|^{\varepsilon}$.

Theorem

Fix $\varepsilon > 0$, and let G be an ε-quasirandom simple group. Then

- *2 random elements give an expander Cayley graph with high probability (Breuillard-Green-Guralnick-Tao '15)*

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The corresponding questions are very open in high rank, or even for $\mathrm{Sym}(n)$.

The world of finite groups

$\mathrm{PSL}_2(p),\ \mathrm{PSU}_3(p)$

$\mathrm{SL}_2(p) \times \mathrm{SL}_2(p),\ (\mathbb{F}_p)^2 \rtimes \mathrm{SL}_2(p),\ \mathrm{Sp}_4(\mathbb{Z}/p^3\mathbb{Z})$

$\vdots$

$\mathrm{Alt}(n),\ \mathrm{PSL}_n(p)$

$\ddots$

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Theorem (S. '26)

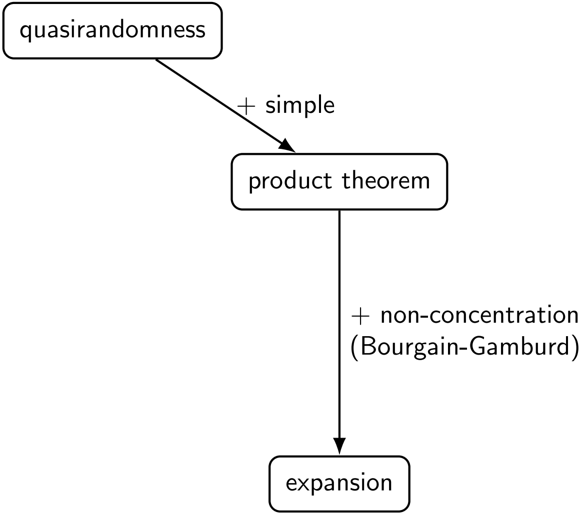
Let G be ε-quasirandom. If $G = \langle S \rangle$, then

$$\operatorname{gap}(G,S) \gg_{\varepsilon,|S|} \min\{\operatorname{gap}(Q,S) : Q \text{ a simple quotient of } G\}.$$

The converse inequality is trivial.

$$\begin{tikzcd} \boxed{G} \arrow[r, dash, "\text{simple}"'] & \boxed{N_1} \arrow[r, dash, "\text{abelian}"'] & \boxed{N_2} \arrow[r, dash, "\text{abelian}"'] & \boxed{N_3} \arrow[r, dashed] & {} \end{tikzcd}$$

What is happening?



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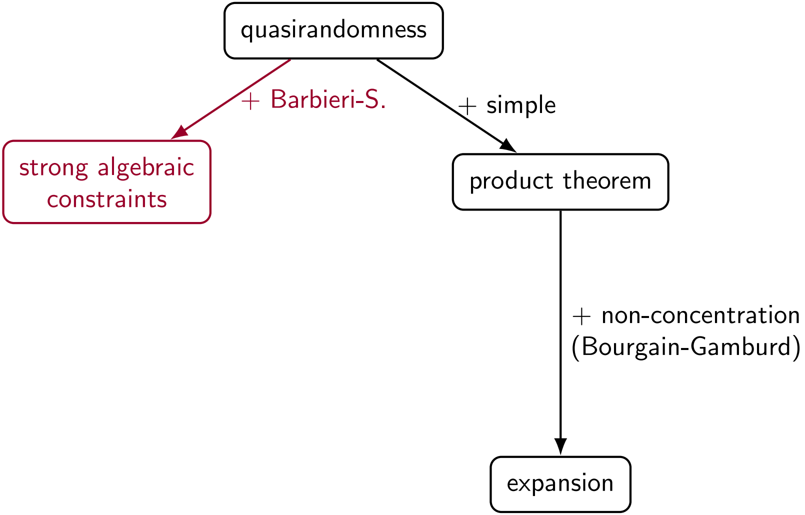

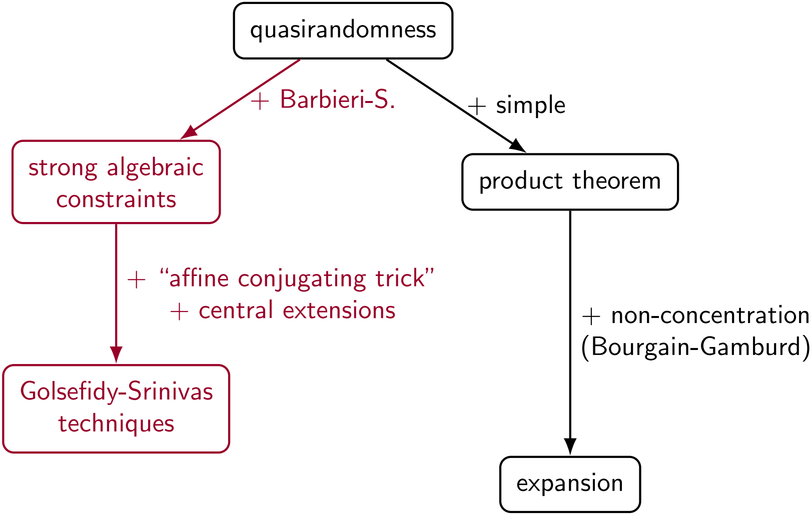
quasirandomness
+ Barbieri-S.
+ simple
strong algebraic constraints
product theorem
+ "affine conjugating trick"
+ central extensions
+ non-concentration (Bourgain-Gamburd)
Golsefidy-Srinivas techniques
expansion

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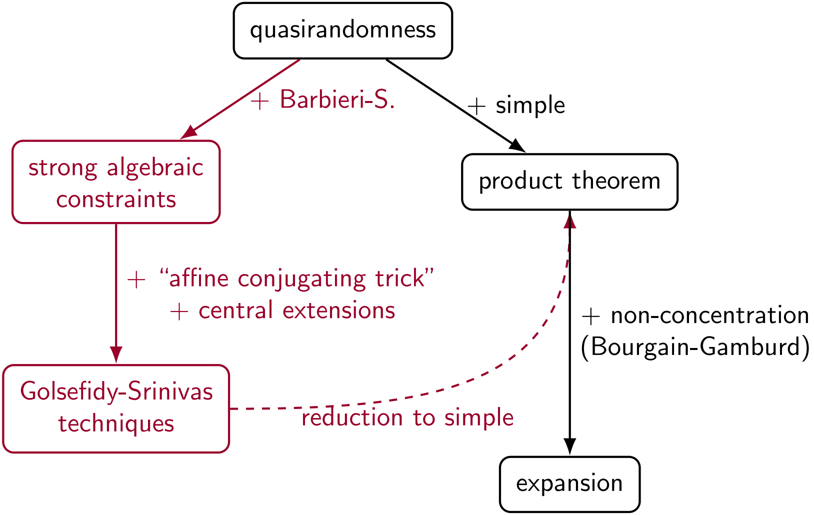

Theorem (Bourgain-Gamburd equivalence, S. '26)

Let $G = \langle S \rangle$ be ε-quasirandom. If there exists $\tau > 0$ such that

$$\sup_{H<G} \mu_{G,S}^{(2n)}(H) \leq |G|^{-\tau}$$

for some $n \leq \tau^{-1} \log |G|$, then $\mathrm{Cay}(G, S)$ is an $\varepsilon'(\varepsilon, \tau)$-expander

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Theorem (S. '26)

If G is ε-quasirandom, then

- $1 + \lfloor \varepsilon^{-1} \rfloor$ *random elements give an expander Cayley graph with high probability*
- $\mathrm{diam}(G) \leq (\log |G|)^{O_\varepsilon(1)}$.

Ingredients: Crown theory, "affine conjugating trick", Ore-like lemma.