Lipschitz Harmonic Functions and Coarse Harmonic Coordinates on Polynomial Growth Groups

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Growth and Expansion in Groups

This talk is based on a joint work with Mayukh Mukherjee and Soumyadip Thandar

Overview

1. Preliminaries
2. Structure of Lipschitz harmonic functions
3. Coarse harmonic coordinates on polynomial growth groups

Preliminaries I

Throughout, G is a finitely generated (infinite) group with a fixed finite symmetric generating set S (used to define the word metric $|\cdot|$), and μ is a probability measure on G.

- μ is *Abelian centered* if $\sum_{g\in G}\mu(g)\phi(g) = 0$ for all $\phi \in \mathrm{Hom}(G,\mathbb{R})$.
- μ is *adapted* (or non-degenerate) if the semigroup generated by $\mathrm{supp}(\mu)$ equals G.
- μ is *smooth* if there exists $\zeta > 0$ such that

$$\Psi(\zeta) := \sum_{x\in G}\mu(x)e^{\zeta|x|} < \infty. \tag{1}$$

- A function $f : G \to \mathbb{C}$ is *μ-harmonic at $k \in G$* if

$$f(k) = \sum_{g\in G}\mu(g)f(kg), \tag{2}$$

  and the series converges absolutely. If (2) holds for all k, we say f is μ-harmonic on G.

Preliminaries II

- For a function $f : G \to \mathbb{C}$ and symmetric generating set S of G, define the (right) gradient by $(\nabla_S f(x))_s := \partial^s f(x) := f(xs) - f(x)$ $(s \in S)$ and the seminorm

$$\|\nabla_S f\|_\infty := \sup_{s\in S} \sup_{x\in G} |\partial^s f(x)|.$$

Denote

$$P^k(G) = \{f : G \to C \mid \partial^{u_1} \dots \partial^{u_{k+1}} f \equiv 0 \ (u_i \in G)\}$$

$$\mathrm{LHF}(G, \mu) = \{f : G \to \mathbb{C} \mid f \text{ is } \mu\text{-harmonic}, \ \|\nabla_S f\|_\infty < \infty\}$$

$$\mathrm{HF}_k(G, \mu) = \{f : G \to \mathbb{C} \mid f \text{ is } \mu\text{-harmonic}, \ |f(x)| \leq C|x|^k \ (x \neq e)\}$$

- If X and Y are metric spaces, a map $\phi : X \to Y$ is called a quasi-isometry if:
  1. there exists $L, C > 0$ such that for all $x_1, x_2 \in X$

$$\frac{1}{L} d(x_1, x_2) - C \leq d(\phi(x_1), \phi(x_2)) \leq L d(x_1, x_2) + C$$

  2. $\phi(X)$ is *coarsely dense* in Y, i.e. there exists $M > 0$ such that for any $y \in Y$ there exists $x \in X$ such that $d(\phi(x), y) \leq M$.

Affine structure of LHF

Alexopoulous (2002): G – polynomial growth, μ – Abelian-centered, finitely supported $\Rightarrow$ polynomially growing harmonic functions are polynomials.

Meyerovitch, Perl, Tointon, Yadin (2016): G–polynomial growth, μ–symmetric, adapted and smooth $\Rightarrow \mathrm{HF}_k$ consists of polynomials of degree atmost k when restricted to some finite index subgroup.

Theorem [Mukherjee, S, Thandar] (Affine rigidity):
Let G be a finitely generated nilpotent group, and let μ be an adapted, smooth, and *Abelian-centered* probability measure on G. Then

$$\mathrm{HF}_1(G,\mu) = \mathsf{LHF}(G,\mu) = P^1(G).$$

Moreover, if G is virtually nilpotent (but not nilpotent) we have $\mathrm{HF}_1(G,\mu) \cong \mathsf{LHF}(G,\mu)$, and they are exactly equal if $\dim \mathrm{HF}_1(G,\mu) < \infty$.

Coarse harmonic coordinates I

Let N and M be finitely generated groups, with Abelianization maps

$$[\cdot]_N : N \to N_{\mathrm{ab}}, \qquad [\cdot]_M : M \to M_{\mathrm{ab}}.$$

Define the projections

$$\pi_N : N \xrightarrow{[\cdot]_N} N_{\mathrm{ab}} \hookrightarrow N_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{R},$$

and similarly

$$\pi_M : M \xrightarrow{[\cdot]_M} M_{\mathrm{ab}} \hookrightarrow M_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Definition: (Bounded Abelian defect) A map $\Psi : N \to M$ has *bounded Abelian defect* if

$$\Delta_{\mathrm{ab}}(\Psi) := \sup_{x,y \in N} \left\| \pi_M(\Psi(xy)) - \pi_M(\Psi(x)) - \pi_M(\Psi(y)) \right\| < \infty.$$

Coarse harmonic coordinates I

Theorem [Mukherjee, S, Thandar] (Coarse harmonic coordinates):
Let N and M be finitely generated nilpotent groups and $\Psi : N \to M$ be a quasi-isometry having bounded Abelian defect. Then, there exists a unique linear isomorphism

$$L_{ab} : N_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{R} \to M_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{R}$$

such that

$$\sup_{x \in N} \left\| \pi_M(\Psi(x)) - L_{ab}(\pi_N(x)) \right\| < \infty. \tag{3}$$

Consequently, we define

$$T_\Psi : \mathrm{Hom}(N_{\mathrm{ab}}, \mathbb{R}) \longrightarrow \mathrm{Hom}(M_{\mathrm{ab}}, \mathbb{R})$$

by

$$T_\Psi(\varphi) := \varphi \circ L_{ab}^{-1} \qquad (\forall \varphi \in \mathrm{Hom}(N_{\mathrm{ab}}, \mathbb{R})).$$

Coarse harmonic coordinates II

Let $r = \dim_{\mathbb{R}} \mathrm{Hom}(N_{\mathrm{ab}}, \mathbb{R})$, fix a basis $\{\varphi_i\}_{i=1}^r$ of $\mathrm{Hom}(N_{\mathrm{ab}}, \mathbb{R})$ and put $\psi_i := T_\Psi(\varphi_i)$ for $i = 1, \ldots, r$. Define coarse harmonic coordinates on N and M by

$$F_N(x) := \big(\varphi_i([x])\big)_{i=1}^r, \qquad F_M(y) := \big(\psi_i([y])\big)_{i=1}^r.$$

Then

$$\sup_{x \in N} \big\| F_M(\Psi(x)) - F_N(x) \big\| < \infty.$$

Coarse harmonic coordinates III

Now let G be a finitely generated group of polynomial growth with a finite-index nilpotent subgroup $N \leq G$. Fix a right coset decomposition

$$G = \bigsqcup_{j=1}^{k} N g_j$$

with $g_1 = e_G$. For $x \in G$, write $x = n g_j$ with $n \in N$ and define

$$F_G(x) := F_N(n).$$

Thus F_G *projects* harmonic coordinates from N to G along the cosets. We refer to F_G as a system of coarse harmonic coordinates on G associated to the nilpotent subgroup N and the basis $\{\varphi_i\}$.

Coarse harmonic coordinates IV

Theorem [Mukherjee, S, Thandar] (Coarse straightening in harmonic coordinates): Let G, H be finitely generated groups of polynomial growth, and let $\Phi : G \to H$ be a quasi-isometry. Let $N \leq G$ and $M \leq H$ be finite-index nilpotent subgroups, and let $\Psi : N \to M$ be a quasi-isometry at a uniformly bounded distance from $\Phi|_N$. Let F_G and F_H be coarse harmonic coordinates constructed using a basis of $\mathrm{Hom}(N_{\mathrm{ab}}, \mathbb{R})$ and its image under T_Ψ. If Ψ has bounded Abelian defect, then

$$\sup_{x \in G} \left\| F_H(\Phi(x)) - F_G(x) \right\| < \infty.$$

THANK YOU!