Abelian structure in non-abelian approximate groups

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Growth and Expansion in Groups

Avoiding 3-term arithmetic progressions

Theorem (Roth '53)

If $A \subset \mathbb{Z}/N\mathbb{Z}$ contains no nontrivial 3-term arithmetic progression, then $|A| = o(N)$.

(A nontrivial 3-AP is a set $\{x, x+y, x+2y\}$ with $y \neq 0$.)

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Question

Let G be an arbitrary finite group. If $A \subset G$ contains no nontrivial 3-AP, must we have $|A| = o(|G|)$?

(A nontrivial 3-AP here is a set $\{x, xy, xy^2\}$ with $y \neq \mathrm{id}_G$.)

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(1) Let A be fairly dense in G. Let $H \leq G$ be abelian from Pyber's theorem.

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The largest density of a 3-AP-free set in $\mathbb{Z}/N\mathbb{Z}$ is quasipolynomial anyway! So this argument loses little quantitatively.

Smaller sets in groups, and growth

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Theorem (S. '25+)

Let G be residually solvable. Let $A \subset G$ satisfy $1 \in A = A^{-1}$. Write $K = |A^3|/|A|$. Then there exists an abelian subgroup $H \leq G$ with

$$|A^4 \cap H| \geq \exp\left(c \frac{\log^{1/6}|A|}{\log 2K} \right).$$

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This result is enough to obtain

Theorem (S. '25+)

If A is a 3-AP-free subset of any residually solvable group, then

$$|A^2| \geq |A| \exp\left(c(\log|A|)^{1/1305}\right).$$

An application to Cayley graphs

Theorem (Conlon–Fox–Pham–Yepremyan '24+)

For some constant $C > 0$, every finite abelian group G with $\gcd(|G|, 6) = 1$ admits a Cayley graph with no clique or independent set of size $C \log_2|G|$.

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Lemma (Conlon–Fox–Pham–Yepremyan '24+)

If G is abelian and $A \subset G$ has no nontrivial solutions to $ab^{-1} = (cd^{-1})^2 = (ef^{-1})^4 = (gh^{-1})^8$, then $|AA^{-1}| \geq |A|^{1.01}$.

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Theorem (S. '25+)

The above theorem holds with the word "abelian" removed.

Extending from solvable groups

In fact, such Cayley graphs exist for all groups of almost all orders.

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One tool:

Theorem (S. '25+)

Let $A \subset \mathrm{GL}_d(\mathbb{F})$ satisfy $1 \in A = A^{-1}$. Write $K = |A^3|/|A|$. Then there exists an abelian subgroup $H \leq \mathrm{GL}_d(\mathbb{F})$ with

$$|A^4 \cap H| \geq \frac{1}{2K}|A|^{\varepsilon_d},$$

for ε_d decaying quasi-polynomially in d.

Stitch these together through the chief series and use CFSG.