Hypergeometric Groups and A Question of Sarnak

Lal Bahadur Sahu

July 7, 2026

Table of contents

Definition

Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}^*$ *with* $a_j \neq b_k$ *for all* $j, k = 1, \ldots, n$*. A hypergeometric group associated to monic polynomials* f *and* g *is a subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *generated by elements*

$$A, B, C \in \mathrm{GL}(n, \mathbb{C})$$

Definition

Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}^*$ *with* $a_j \neq b_k$ *for all* $j, k = 1, \ldots, n$. *A hypergeometric group associated to monic polynomials* f *and* g *is a subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *generated by elements*

$$A, B, C \in \mathrm{GL}(n, \mathbb{C})$$

where A, B *are companion matrices of* f, g *respectively, such that*

$$C = A^{-1}B$$

and

$$\det(xI_n - A) = \prod_{j=1}^{n}(x - a_j), \det(xI_n - B) = \prod_{j=1}^{n}(x - b_j)$$

and C *is a transvection, i.e.,* $\mathrm{rank}(C - I_n) = 1$.

Zariski Closure of Hypergeometric Groups

We consider cases where f and g have integral coefficients and constant terms ± 1. In addition, we assume that f, g form a primitive pair, are self-reciprocal, and do not have common root. In 1989, Beukers and Heckman have determined the Zariski closure G of $\Gamma(f, g)$ as follows:

Zariski Closure of Hypergeometric Groups

We consider cases where f and g have integral coefficients and constant terms ± 1. In addition, we assume that f, g form a primitive pair, are self-reciprocal, and do not have common root. In 1989, Beukers and Heckman have determined the Zariski closure G of $\Gamma(f, g)$ as follows:

1. If n is even, $\Gamma(f, g)$ is infinite and $\frac{f(0)}{g(0)} = 1$, then $\Gamma(f, g)$ preserves a non-degenerate integral symplectic form Ω on $\mathbb{Z}^n$ and $\mathrm{G} = \mathrm{Sp}_{\Omega}$.

Zariski Closure of Hypergeometric Groups

We consider cases where f and g have integral coefficients and constant terms ± 1. In addition, we assume that f, g form a primitive pair, are self-reciprocal, and do not have common root. In 1989, Beukers and Heckman have determined the Zariski closure G of $\Gamma(f, g)$ as follows:

1. If n is even, $\Gamma(f, g)$ is infinite and $\frac{f(0)}{g(0)} = 1$, then $\Gamma(f, g)$ preserves a non-degenerate integral symplectic form Ω on $\mathbb{Z}^n$ and $\mathrm{G} = \mathrm{Sp}_{\Omega}$.
2. If $\Gamma(f, g)$ is infinite and $\frac{f(0)}{g(0)} = -1$, then $\Gamma(f, g)$ preserves a non-degenerate quadratic form Q on $\mathbb{Z}^n$ and $\mathrm{G} = \mathrm{O}_{\mathrm{Q}}$.

Zariski Closure of Hypergeometric Groups

We consider cases where f and g have integral coefficients and constant terms ± 1. In addition, we assume that f, g form a primitive pair, are self-reciprocal, and do not have common root. In 1989, Beukers and Heckman have determined the Zariski closure G of $\Gamma(f, g)$ as follows:

1. If n is even, $\Gamma(f, g)$ is infinite and $\frac{f(0)}{g(0)} = 1$, then $\Gamma(f, g)$ preserves a non-degenerate integral symplectic form Ω on $\mathbb{Z}^n$ and $\mathrm{G} = \mathrm{Sp}_\Omega$.
2. If $\Gamma(f, g)$ is infinite and $\frac{f(0)}{g(0)} = -1$, then $\Gamma(f, g)$ preserves a non-degenerate quadratic form Q on $\mathbb{Z}^n$ and $\mathrm{G} = \mathrm{O}_\mathrm{Q}$.
3. $\Gamma(f, g)$ is *finite* if and only if either $\alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n$ or $\beta_1 < \alpha_1 < \cdots < \beta_n < \alpha_n$, where $a_j = e^{2\pi i \alpha_j}$ and $b_k = e^{2\pi i \beta_k}$, in addition with $\alpha_j, \beta_k \in [0, 1)$.

Definition

A hypergeometric group $\Gamma(f,g) \subseteq \mathrm{G}(\mathbb{Z})$ is called arithmetic if it has finite index in $\mathrm{G}(\mathbb{Z})$; and thin, otherwise, where G is the Zariski closure of $\Gamma(f,g)$ inside $\mathrm{GL}_n(\mathbb{C})$.

Definition

A hypergeometric group $\Gamma(f,g) \subseteq \mathrm{G}(\mathbb{Z})$ is called arithmetic if it has finite index in $\mathrm{G}(\mathbb{Z})$; and thin, otherwise, where G is the Zariski closure of $\Gamma(f,g)$ inside $\mathrm{GL}_n(\mathbb{C})$.

The following question earned great interest of many mathematicians after Sarnak asked it:

Definition

A hypergeometric group $\Gamma(f,g) \subseteq \mathrm{G}(\mathbb{Z})$ *is called arithmetic if it has finite index in* $\mathrm{G}(\mathbb{Z})$*; and thin, otherwise, where* G *is the Zariski closure of* $\Gamma(f,g)$ *inside* $\mathrm{GL}_n(\mathbb{C})$*.*

The following question earned great interest of many mathematicians after Sarnak asked it:

Question

Determine polynomial pairs (f,g) *for which the associated hypergeometric groups* $\Gamma(f,g)$ *are arithmetic or thin.*

Singh-Venkataramana Criterion

Theorem (2014)

Let $n \geq 4$. Suppose f, g satisfy the hypotheses of the symplectic case of Beukers and Heckman. Let A, B be companion matrices of polynomials f, g respectively and $\Gamma(f, g) = \langle A, B \rangle \subseteq \mathrm{SL}_n(\mathbb{Z})$.

Singh-Venkataramana Criterion

Theorem (2014)

Let $n \geq 4$. Suppose f, g satisfy the hypotheses of the symplectic case of Beukers and Heckman. Let A, B be companion matrices of polynomials f, g respectively and $\Gamma(f,g) = \langle A, B \rangle \subseteq \mathrm{SL}_n(\mathbb{Z})$. Assume in addition, that the leading coefficient c of $f - g$ satisfies

$$\mid c \mid \leq 2.$$

Then the hypergeometric group $\Gamma(f,g)$ is an arithmetic subgroup of the corresponding symplectic group.

Literature of $\mathrm{Sp}(4)$

There are 111 pairs of degree four product of cyclotomic polynomials, determined by conditions of Beukers and Heckman in the symplectic case, whose associated hypergeometric group's arithmeticity or thinness has been settled by following:

Literature of Sp(4)

There are 111 pairs of degree four product of cyclotomic polynomials, determined by conditions of Beukers and Heckman in the symplectic case, whose associated hypergeometric group's arithmeticity or thinness has been settled by following:

1. Arithmeticity of 60 pairs follows from the sufficient criterion of Singh-Venkataramana. In the same article, 4 others have been proved to be arithmetic which does not satisfy sufficient criterion.

Literature of $\mathrm{Sp}(4)$

There are 111 pairs of degree four product of cyclotomic polynomials, determined by conditions of Beukers and Heckman in the symplectic case, whose associated hypergeometric group's arithmeticity or thinness has been settled by following:

1. Arithmeticity of 60 pairs follows from the sufficient criterion of Singh-Venkataramana. In the same article, 4 others have been proved to be arithmetic which does not satisfy sufficient criterion.
2. It follows from Brav, Thomas (2014) that 13 of them are thin.

Literature of Sp(4)

There are 111 pairs of degree four product of cyclotomic polynomials, determined by conditions of Beukers and Heckman in the symplectic case, whose associated hypergeometric group's arithmeticity or thinness has been settled by following:

1. Arithmeticity of 60 pairs follows from the sufficient criterion of Singh-Venkataramana. In the same article, 4 others have been proved to be arithmetic which does not satisfy sufficient criterion.
2. It follows from Brav, Thomas (2014) that 13 of them are thin.
3. Singh (2015) and Singh (2017) proved 23 more pairs are arithmetic.

Literature of $\mathrm{Sp}(4)$

There are 111 pairs of degree four product of cyclotomic polynomials, determined by conditions of Beukers and Heckman in the symplectic case, whose associated hypergeometric group's arithmeticity or thinness has been settled by following:

1. Arithmeticity of 60 pairs follows from the sufficient criterion of Singh-Venkataramana. In the same article, 4 others have been proved to be arithmetic which does not satisfy sufficient criterion.
2. It follows from Brav, Thomas (2014) that 13 of them are thin.
3. Singh (2015) and Singh (2017) proved 23 more pairs are arithmetic.
4. Bajpai, Singh, Singh (2023) proved 1 more to be arithmetic.

Literature of $\mathrm{Sp}(4)$

There are 111 pairs of degree four product of cyclotomic polynomials, determined by conditions of Beukers and Heckman in the symplectic case, whose associated hypergeometric group's arithmeticity or thinness has been settled by following:

1. Arithmeticity of 60 pairs follows from the sufficient criterion of Singh-Venkataramana. In the same article, 4 others have been proved to be arithmetic which does not satisfy sufficient criterion.
2. It follows from Brav, Thomas (2014) that 13 of them are thin.
3. Singh (2015) and Singh (2017) proved 23 more pairs are arithmetic.
4. Bajpai, Singh, Singh (2023) proved 1 more to be arithmetic.
5. Bajpai, Dona, Nitsche (2025) proved 5 of the left 10 to be thin. Bajpai, Dona, Nitsche (2026) settled the problem by proving rest 5 to be arithmetic.

Some examples of arithmetic pair (f_1, g_1) and thin pair (f_2, g_2) whose monodromy is associated to Calabi-Yau threefolds

$$f_1(x) = (x-1)^4, \quad g_1(x) = (x^2 - x + 1)^2$$

$$f_2(x) = (x-1)^4, \quad g_2(x) = x^4 + x^3 + x^2 + x + 1$$

Some examples of arithmetic pair (f_1, g_1) and thin pair (f_2, g_2) whose monodromy is associated to Calabi-Yau threefolds

$$f_1(x) = (x-1)^4, \quad g_1(x) = (x^2 - x + 1)^2$$

$$f_2(x) = (x-1)^4, \quad g_2(x) = x^4 + x^3 + x^2 + x + 1$$

Conjecture (Sarnak, 2014)

Hypergeometric group $\Gamma(f, g)$ *for* $n \geq 4$ *even,*

$$f(x) = (x-1)^n, g(x) = x^n + x^{n-1} + \ldots + 1,$$

is conjectured to be thin.

Literature of $\mathrm{O}(5)$

We note some progress in the Orthogonal case:

1. Fuchs, Meiri and Sarnak (2014) produced 8 infinite families of hypergeometric groups in $\mathrm{O}(4,1)$ to be thin.

Literature of $\mathrm{O}(5)$

We note some progress in the Orthogonal case:

1. Fuchs, Meiri and Sarnak (2014) produced 8 infinite families of hypergeometric groups in $\mathrm{O}(4,1)$ to be thin.
2. Venkataramana (2017) produced 11 infinite families, the first examples of higher rank arithmetic hypergeometric subgroups of orthogonal type.

Literature of $\mathrm{O}(5)$

We note some progress in the Orthogonal case:

1. Fuchs, Meiri and Sarnak (2014) produced 8 infinite families of hypergeometric groups in $\mathrm{O}(4,1)$ to be thin.
2. Venkataramana (2017) produced 11 infinite families, the first examples of higher rank arithmetic hypergeometric subgroups of orthogonal type.
3. Out of 73 pairs in $\mathrm{O}(5)$, satisfying the condition of infinite orthogonal type, 7 pairs in $\mathrm{O}(3,2)$ and 1 in $\mathrm{O}(4,1)$ is left to prove, others have been done by Singh (2015), Bajpai and Singh (2017), Bajpai, Singh, Singh (2023), Bajpai and Nitsche (2024).

Literature of $\mathrm{O}(5)$

We note some progress in the Orthogonal case:

1. Fuchs, Meiri and Sarnak (2014) produced 8 infinite families of hypergeometric groups in $\mathrm{O}(4,1)$ to be thin.
2. Venkataramana (2017) produced 11 infinite families, the first examples of higher rank arithmetic hypergeometric subgroups of orthogonal type.
3. Out of 73 pairs in $\mathrm{O}(5)$, satisfying the condition of infinite orthogonal type, 7 pairs in $\mathrm{O}(3,2)$ and 1 in $\mathrm{O}(4,1)$ is left to prove, others have been done by Singh (2015), Bajpai and Singh (2017), Bajpai, Singh, Singh (2023), Bajpai and Nitsche (2024).

Question

Can a sufficient criterion to prove arithmeticity in orthogonal case be given as given in the case of symplectic?

Literature of $\mathrm{Sp}(6)$

Out of 458 pairs in $\mathrm{Sp}(6)$, product of cyclotomic polynomials, satisfying the condition of symplectic case, all but three cases is settled by Singh and Venkataramana (2014), Bajpai, Dona, Singh, Singh (2021), Detinko, Flannery, Hulpke (2023), Bajpai, Dona, Nitsche (2025), Bajpai, Dona, Nitsche (2026). The recent work in May 2026 by Riestenberg, Taha, Trettel, Wienhard in arxiv claims to prove 2 more to be arithmetic leaving only 1 case unresolved.

A research talk is incomplete until you state one of your own theorem; here is the one from me which I got after submitting my abstract for this Gong Show.

Theorem (Sahu, 2026)

There are at least eight infinite families of arithmetic symplectic hypergeometric groups that do not satisfy the criterion of Singh and Venkataramana.

Theorem (Sahu, 2026)

There are at least eight infinite families of arithmetic symplectic hypergeometric groups that do not satisfy the criterion of Singh and Venkataramana.

For example, the hypergeometric groups associated to the pairs $f(x) = (x-1)^4 P(x^{10})$ and $g(x) = (x^4 + x^2 + 1)Q(x^{10})$, where P, Q are self-reciprocals, monic polynomials of degree m with integral coefficients, in addition with $P(0) = Q(0) = 1$ such that f, g are coprime, are arithmetic subgroups of $\mathrm{Sp}_{4+10m}(\mathbb{Z})$.

Theorem (Sahu, 2026)

There are at least eight infinite families of arithmetic symplectic hypergeometric groups that do not satisfy the criterion of Singh and Venkataramana.

For example, the hypergeometric groups associated to the pairs $f(x) = (x-1)^4 P(x^{10})$ and $g(x) = (x^4 + x^2 + 1)Q(x^{10})$, where P, Q are self-reciprocals, monic polynomials of degree m with integral coefficients, in addition with $P(0) = Q(0) = 1$ such that f, g are coprime, are arithmetic subgroups of $\mathrm{Sp}_{4+10m}(\mathbb{Z})$.

As far as I know it provides the first examples of an infinite family of arithmetic $\Gamma(f, g)$ where f, g DO NOT satisfy the criterion of Singh and Venkataramana.

Thank You!
Questions?