Cores of Modules over Free Group Algebras

Danielle Ernst–West

Joint work with Doron Puder

Free group algebras and their ideals

Definition

Fix a field K and a free group F. The **free group algebra** KF is

$$KF := \left\{ \sum_{u \in F} \alpha_u \, u \;\middle|\; \alpha_u \in K, \text{ finite support} \right\}.$$

Free group algebras and their ideals

Definition

Fix a field K and a free group F. The **free group algebra** KF is

$$KF := \left\{ \sum_{u \in F} \alpha_u \, u \;\middle|\; \alpha_u \in K, \text{ finite support} \right\}.$$

Theorem (Cohn '64, Lewin '69)

Every (right) ideal of KF is a free KF-module.

Initial motivation — word measures

Fix $1 \neq w \in F$.

$$\begin{tikzcd} \boxed{\begin{array}{c} w\text{-measure} \\ \text{on } S_n \end{array}} \arrow[r, "\text{understand}"] & \boxed{\begin{array}{c} \text{subgroups} \\ w \in H \leq F \end{array}} \end{tikzcd}$$

$$\begin{tikzcd} \boxed{\begin{array}{c} w\text{-measure} \\ \text{on } \mathrm{GL}_n(\mathbb{F}_q) \end{array}} \arrow[r, "\text{understand}"] & \boxed{\begin{array}{c} \text{ideals} \\ w - 1 \in I \leq \mathbb{F}_q F \end{array}} \end{tikzcd}$$

Interesting problems

Fix $f \in KF$ and finitely generated ideals $I, J \leq KF$.

Interesting problems

Fix $f \in KF$ and finitely generated ideals $I, J \leq KF$.

1. **Constructing free bases.** Given a set of generators of I, can one find a basis of I?

Interesting problems

Fix $f \in KF$ and finitely generated ideals $I, J \leq KF$.

1. **Constructing free bases.** Given a set of generators of I, can one find a basis of I?
2. **Membership and inclusion.** Is $f \in I$? Is $I \leq J$?

Interesting problems

Fix $f \in KF$ and finitely generated ideals $I, J \leq KF$.

1. **Constructing free bases.** Given a set of generators of I, can one find a basis of I?
2. **Membership and inclusion.** Is $f \in I$? Is $I \leq J$?
3. **Intersections.** What is $I \cap J$? Is $I \cap J$ necessarily finitely generated?

Interesting problems

Fix $f \in KF$ and finitely generated ideals $I, J \leq KF$.

1. **Constructing free bases.** Given a set of generators of I, can one find a basis of I?
2. **Membership and inclusion.** Is $f \in I$? Is $I \leq J$?
3. **Intersections.** What is $I \cap J$? Is $I \cap J$ necessarily finitely generated?
4. **Primitivity testing and free factors.**
   - Assume $f \in I$. Is f *primitive* in I? *(is f a basis element of I?)*
   - Assume $I \leq J$. Is I a free factor of J? *(can a basis of I be completed to a basis of J?)*

Stallings' core graphs

$$H = \langle\, a,\ b^3,\ bab^{-1} \,\rangle \leq F(a,b)$$

Stallings' core graphs

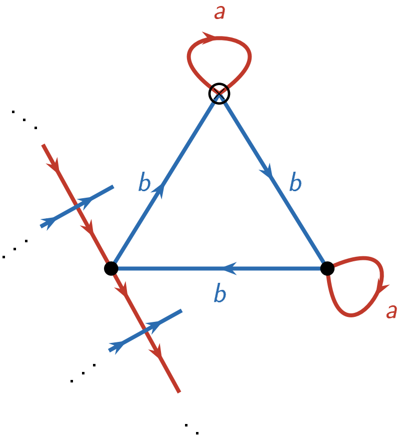
$$H = \langle\, a,\ b^3,\ bab^{-1}\,\rangle \leq F(a,b)$$

$$\mathrm{Sch}(F \curvearrowright F/H, \{a,b\})$$

Stallings' core graphs

$$H = \langle\, a,\ b^3,\ bab^{-1}\,\rangle \leq F(a,b)$$

$$\mathrm{Sch}(F \curvearrowright F/H, \{a,b\})$$



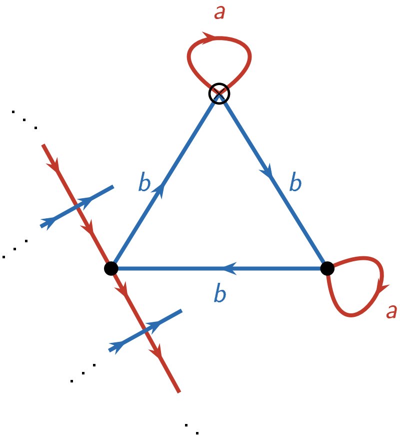
Stallings' core graphs

$$H = \langle\, a,\ b^3,\ bab^{-1} \,\rangle \leq F(a,b)$$

$$\mathrm{Sch}(F \curvearrowright F/H, \{a,b\})$$

the core graph of H:

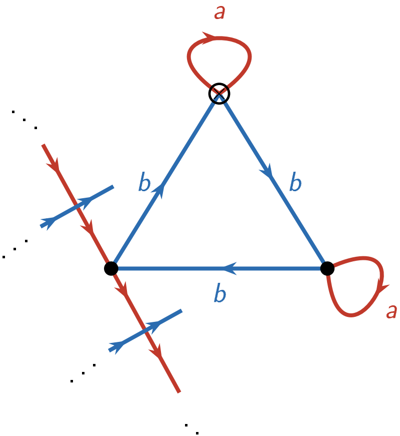
Γ_H = Schreier graph without hanging trees



Stallings' core graphs

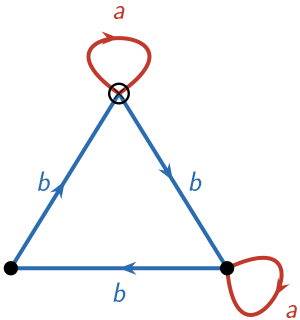
$H = \langle\, a,\ b^3,\ bab^{-1} \,\rangle \leq F(a, b)$

$\mathrm{Sch}(F \curvearrowright F/H, \{a, b\})$



the core graph of H:

Γ_H = Schreier graph without hanging trees



Motto

"The core graph Γ_H is unique and contains all of the important information about the subgroup H."

From subgroups to ideals

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{understand}\\ H \ \leq \ F\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{study}\\ F \ \curvearrowright \ F/H\end{array}} \end{tikzcd}$$

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{understand}\\ I \ \leq \ KF\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{study}\\ {}^{KF}\!/_{I} \text{ as a } KF\text{-module}\end{array}} \end{tikzcd}$$

Pre-cores of modules over KF

Fix a basis B of F.

Let M be a f.p. KF-module. E.g. $M = KF/I$, with $I \leq KF$ a f.g. ideal.

Pre-cores of modules over KF

Fix a basis B of F.

Let M be a f.p. KF-module. E.g. $M = KF/I$, with $I \leq KF$ a f.g. ideal.

Definition (EW–Puder 26$^+$)

Let $V \leq M$ be a finite-dimensional K-subspace of M. V is a **pre-core** of M if

$$M \;\cong\; {}^{V\,KF}\!/_{R(V)\,KF}.$$

Motto

"A pre-core of a f.p. module M is a finite-dimensional vector space $V \leq M$ containing all the important information about M."

Canonical core

Definition (EW–Puder 26$^+$)

The **canonical core** of M, denoted $\mathcal{C}(M)$, is the intersection of all pre-cores of M.

Canonical core

Definition (EW–Puder 26^+)

The **canonical core** of M, denoted $\mathcal{C}(M)$, is the intersection of all pre-cores of M.

Theorem (EW–Puder 26^+)

Given a pre-core of a finitely presented module M, trim the "hanging trees" to obtain the core $\mathcal{C}(M)$.

Applications

Prop. (EW–Puder 26$^+$)

$$I \leq J \iff \text{there exists a core morphism } \mathcal{C}(KF/I) \to \mathcal{C}(KF/J).$$

Applications

Prop. (EW–Puder 26$^+$)

$$I \leq J \quad \Longleftrightarrow \quad \text{there exists a core morphism } \mathcal{C}(KF/I) \to \mathcal{C}(KF/J).$$

Theorem (EW–Puder 26$^+$)

I is a free factor of J $\quad \Longleftrightarrow \quad$ there exists a core morphism $\Psi \colon \mathcal{C}(KF/I) \to \mathcal{C}(KF/J)$ and $\operatorname{Trim}(\ker \Psi) = 0$.

Thank you!

"Proof"

(∗) I is a free factor of $J \iff I \leq J$ and J/I is a free KF-module.

(∗)

$$\begin{tikzcd}
\ker \Psi \arrow[d, dotted] & \Psi \colon \mathcal{C}(KF/I) \arrow[r] \arrow[d, dotted, "\text{lifts}"'] & \mathcal{C}(KF/J) \arrow[d, dotted] \\
\ker \widetilde{\Psi} = J/I & \widetilde{\Psi} \colon \quad KF/I \arrow[r] & KF/J
\end{tikzcd}$$

(∗) $\mathcal{C}(J/I) = \mathrm{Trim}(\ker \Psi)$.

(∗) a KF-module M is free $\iff \mathcal{C}(M) = 0$.