Roots $x_k(y)$ of a formal power series $$f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n,$$ with applications to graph enumeration and q-series

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References:

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`http://www.maths.qmw.ac.uk/~pjc/csgnotes/sokal/`

2. The leading root of the partial theta function, arXiv:1106.1003 [math.CO], Adv. Math. **229**, 2603–2621 (2012).

The deformed exponential function $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$

- Defined for complex x and y satisfying $|y| \le 1$
- Analytic in $\mathbb{C} \times \mathbb{D}$, continuous in $\mathbb{C} \times \overline{\mathbb{D}}$
- $F(\,\cdot\,, y)$ is entire for each $y \in \overline{\mathbb{D}}$
- Valiron (1938): "from a certain viewpoint the simplest entire function after the exponential function"

Applications:

- Statistical mechanics: Partition function of one-site lattice gas
- Combinatorics: Enumeration of connected graphs,
  generating function for Tutte polynomials on K_n
  (also acyclic digraphs, inversions of trees, ...)
- Functional-differential equation: $F'(x) = F(yx)$ where $' = \partial/\partial x$
- Complex analysis: Whittaker and Goncharov constants

Application to enumeration of connected graphs

- Let $a_{n,m} = \#$ graphs with n labelled vertices and m edges
- Generating polynomial $A_n(v) = \sum\limits_m a_{n,m}\, v^m$
- Exponential generating function $A(x,v) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} A_n(v)$
- Of course $a_{n,m} = \binom{n(n-1)/2}{m} \quad \Longrightarrow \quad A_n(v) = (1+v)^{n(n-1)/2} \quad \Longrightarrow$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} A_n(v) \;=\; F(x, 1+v)$$

- Now let $c_{n,m} = \#$ *connected* graphs with n labelled vertices and m edges
- Generating polynomial $C_n(v) = \sum\limits_m c_{n,m}\, v^m$
- Exponential generating function $C(x,v) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n!} C_n(v)$
- No simple explicit formula for $C_n(v)$ is known, but ...
- The *exponential formula* tells us that $C(x,v) = \log A(x,v)$, i.e.

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) \;=\; \log F(x, 1+v)$$

  [see Tutte (1967) and Scott–A.D.S., arXiv:0803.1477 for generalizations to the Tutte polynomials of the complete graphs K_n]
- Usually considered as formal power series
- But series are *convergent* if $|1+v| \le 1$
  [see also Flajolet–Salvy–Schaeffer (2004)]

Elementary analytic properties of $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$

- $\boldsymbol{y = 0}$: $F(x,0) = 1 + x$

- $\boldsymbol{0 < |y| < 1}$: $F(\,\cdot\,, y)$ is a nonpolynomial entire function of order 0:
$$F(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \;-\; \frac{x}{x_k(y)}\right)$$
where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 0$

- $\boldsymbol{y = 1}$: $F(x,1) = e^x$

- $\boldsymbol{|y| = 1}$ **with** $\boldsymbol{y \neq 1}$: $F(\,\cdot\,, y)$ is an entire function of order 1 and type 1:
$$F(x,y) \;=\; e^x \prod_{k=0}^{\infty} \left(1 \;-\; \frac{x}{x_k(y)}\right) e^{x/x_k(y)} \;.$$
where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 1$

  [see also Ålander (1914) for y a root of unity; Valiron (1938) and Eremenko–Ostrovskii (2007) for y not a root of unity]

- $\boldsymbol{|y| > 1}$: The series $F(\,\cdot\,, y)$ has radius of convergence 0

Consequences for $C_n(v)$

- Make change of variables $y = 1 + v$:

$$\overline{C}_n(y) \;=\; C_n(y-1)$$

- Then for $|y| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \overline{C}_n(y) \;=\; \log F(x,y) \;=\; \sum_k \log\Bigl(1 \,-\, \frac{x}{x_k(y)}\Bigr)$$

  and hence

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n} \qquad \text{for all } n \ge 1$$

  (also holds for $n \ge 2$ when $|y| = 1$)

- This is a *convergent* expansion for $\overline{C}_n(y)$
- In particular, gives large-n asymptotic behavior

$$\overline{C}_n(y) \;=\; -(n-1)!\, x_0(y)^{-n} \left[1 \,+\, O(e^{-\epsilon n})\right]$$

  whenever $F(\,\cdot\,, y)$ has a unique root $x_0(y)$ of minimum modulus

Question: What can we say about the roots $x_k(y)$?

Small-y expansion of roots $x_k(y)$

- For small $|y|$, we have $F(x,y) = 1 + x + O(y)$, so we expect a convergent expansion

$$x_0(y) \;=\; -1 \;-\; \sum_{n=1}^{\infty} a_n y^n$$

  (easy proof using Rouché: valid for $|y| \lesssim 0.441755$)

- More generally, for each integer $k \geq 0$, write $x = \xi y^{-k}$ and study

$$F_k(\xi, y) \;=\; y^{k(k+1)/2} F(\xi y^{-k}, y) \;=\; \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, y^{(n-k)(n-k-1)/2}$$

  Sum is dominated by terms $n = k$ and $n = k+1$; gives root

$$x_k(y) \;=\; -(k+1) y^{-k} \left[1 + \sum_{n=1}^{\infty} a_n^{(k)} y^n \right]$$

  Rouché argument valid for $|y| \lesssim 0.207875$ uniformly in k:
  all roots are simple and given by convergent expansion $x_k(y)$

- Can also use theta function in Rouché (Eremenko)

Might these series converge for all $|y| < 1$?

Two ways that $x_k(y)$ could fail to be analytic for $|y| < 1$:

1. Collision of roots ($\to$ branch point)
2. Root escaping to infinity

Theorem (Eremenko): No root can escape to infinity for y in the open unit disc $\mathbb{D}$ (except of course at $y = 0$).

In fact, for any compact subset $K \subset \mathbb{D}$ and any $\epsilon > 0$, there exists an integer k_0 such that for all $y \in K \setminus \{0\}$ we have:

(a) The function $F(\,\cdot\,, y)$ has exactly k_0 zeros (counting multiplicity) in the disc $|x| < k_0 |y|^{-(k_0 - \frac{1}{2})}$, and

(b) In the region $|x| \geq k_0 |y|^{-(k_0 - \frac{1}{2})}$, the function $F(\,\cdot\,, y)$ has a simple zero within a factor $1 + \epsilon$ of $-(k+1)y^{-k}$ for each $k \geq k_0$, and no other zeros.

- Proof is based on comparison with a theta function (whose roots are known by virtue of Jacobi's product formula)
- *Conjecture* that roots cannot escape to infinity even in the *closed* unit disc except at $y = 1$

Big Conjecture #1. All roots of $F(\,\cdot\,, y)$ are simple for $|y| < 1$. [and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #1. Each root $x_k(y)$ is analytic in $|y| < 1$.

But I conjecture more ...

Big Conjecture #2. The roots of $F(\,\cdot\,, y)$ are non-crossing *in modulus* for $|y| < 1$:

$$|x_0(y)| \;<\; |x_1(y)| \;<\; |x_2(y)| \;<\; \ldots$$

[and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #2. The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| \;<\; |y|\,|x_{k+1}(y)| \qquad \text{for all } k \geq 0$$

PROOF. Apply the Schwarz lemma to $x_k(y)/x_{k+1}(y)$.

Consequence for the zeros of $\overline{C}_n(y)$

Recall

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$$

and use a variant of the Beraha–Kahane–Weiss theorem [A.D.S., arXiv:cond-mat/0012369, Theorem 3.2] $\implies$ the limit points of zeros of $\overline{C}_n$ are the values y for which the zero of minimum modulus of $F(\,\cdot\,, y)$ is *nonunique*.

So if $F(\,\cdot\,, y)$ has a *unique* zero of minimum modulus for all $y \in \mathbb{D}$ (a weakened form of Big Conjecture #2), then the zeros of $\overline{C}_n$ do not accumulate anywhere in the open unit disc.

I actually conjecture more (based on computations up to $n \approx 80$):

Big Conjecture #3. For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$. [and, I suspect, no zeros with $|y| = 1$ except the point $y = 1$]

What is the evidence for these conjectures?

Evidence #1: Behavior at real y.

Theorem (Laguerre): For $0 \le y < 1$, all the roots of $F(\,\cdot\,, y)$ are simple and negative real.

Corollary: Each root $x_k(y)$ is analytic in a complex neighborhood of the interval $[0, 1)$.

[Real-variables methods give further information about the roots $x_k(y)$ for $0 \le y < 1$: see Langley (2000).]

Now combine this with

Evidence #2: From numerical computation of the series $x_k(y)$ ... [algorithms to be discussed later]

Let MATHEMATICA run for a weekend ...

$$\begin{aligned}
-x_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\
& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
& + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\
& + \ldots + \text{terms through order } y^{899}
\end{aligned}$$

and all the coefficients (so far) are nonnegative!

- Very recently I have computed $x_0(y)$ through order y^{16383}.
- I also have shorter series for $x_k(y)$ for $k \geq 1$.

Big Conjecture #4. For each k, the series $-x_k(y)$ has all nonnegative coefficients.

Combine this with the known analyticity for $0 \leq y < 1$, and Pringsheim gives:

Consequence of Big Conjecture #4. Each root $x_k(y)$ is analytic in the open unit disc.

NEED TO DO: Extended computations for $k = 1, 2, \ldots$ and for symbolic k.

But more is true ...

Look at the *reciprocal* of $x_0(y)$:

$$
\begin{aligned}
-\frac{1}{x_0(y)} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
& - \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\
& - \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\
& - \ldots - \text{terms through order } y^{899}
\end{aligned}
$$

and all the coefficients (so far) beyond the constant term are *nonpositive*!

Big Conjecture #5. For each k, the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

[This implies the preceding conjecture, but is stronger.]

- Relative simplicity of the coefficients of $-1/x_0(y)$ compared to those of $-x_0(y)$ $\longrightarrow$ simpler combinatorial interpretation?
- Note that $x_k(y) \to -\infty$ as $y \uparrow 1$ (this is fairly easy to prove). So $1/x_k(y) \to 0$. Therefore:

Consequence of Big Conjecture #5. For each k, the coefficients (after the constant term) in the series $-(k+1)y^{-k}/x_k(y)$ are the *probabilities* for a positive-integer-valued random variable.

What might such a random variable be???
Could this approach be used to *prove* Big Conjecture #5?

AGAIN NEED TO DO: Extended computations for $k = 1, 2, \ldots$ and for symbolic k.

But I conjecture that even more is true ...

Define $D_n(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ and recall that $-x_0(y) = \lim_{n\to\infty} D_n(y)^{-1/n}$

Big Conjecture #6. For each n,

(a) the series $D_n(y)^{-1/n}$ has all nonnegative coefficients,

and even more strongly,

(b) the series $D_n(y)^{1/n}$ has all nonpositive coefficients after the constant term 1.

Since $D_n(y) > 0$ for $0 \le y < 1$, Pringsheim shows that Big Conjecture #6a implies Big Conjecture #3:

For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$.

Moreover, Big Conjecture #6b $\Longrightarrow$ for each n, the coefficients (after the constant term) in the series $D_n(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.

Such a random variable would generalize the one for $-1/x_0(y)$ in roughly the same way that the binomial generalizes the Poisson.

When stumped, generalize . . . !

Consider a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$$

normalized to $\alpha_0 = \alpha_1 = 1$, or more generally

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y) \, x^n$$

where

(a) $a_0(0) = a_1(0) = 1$;

(b) $a_n(0) = 0$ for $n \ge 2$; and

(c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$.

Examples:

- The “partial theta function”

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2}$$

- The “deformed exponential function”

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$$

- More generally, consider

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n \, y^{n(n-1)/2}}{(1+q)(1+q+q^2) \, \cdots \, (1+q+\ldots+q^{n-1})}$$

  which reduces to Θ_0 when $q=0$, and to F when $q=1$.

- “Deformed binomial” and “deformed hypergeometric” series (see below).

A general approach to the leading root $x_0(y)$

- Start from a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n$$

  where

  (a) $a_0(0) = a_1(0) = 1$
  (b) $a_n(0) = 0$ for $n \ge 2$
  (c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$

  and coefficients lie in a commutative ring-with-identity-element R.

- By (c), each power of y is multiplied by only *finitely many* powers of x.

- That is, f is a formal power series in y whose coefficients are *polynomials* in x, i.e. $f \in R[x][[y]]$.

- Hence, for *any* formal power series $X(y)$ with coefficients in R [not necessarily with zero constant term], the composition $f(X(y), y)$ makes sense as a formal power series in y.

- Not hard to see (by the implicit function theorem for formal power series or by a direct inductive argument) that there exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$.

- We call $x_0(y)$ the **leading root** of f.

- Since $x_0(y)$ has constant term -1, we will write $x_0(y) = -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.

How to compute $\xi_0(y)$?

1. **Elementary method:** Insert $\xi_0(y) = 1 + \sum_{n=1}^{\infty} b_n y^n$ into $f(-\xi_0(y), y) = 0$ and solve term-by-term.

2. Method based on the explicit implicit function formula (see below).

3. Method based on the exponential formula and expansion of $\log f(x, y)$.

- Method #3 is computationally very efficient. (It's what I used above.)

- Method #2 gives an *explicit* formula for the coefficients of $\xi_0(y)$ ...
- Can it also be used to give *proofs*?

The explicit implicit function formula

- See A.D.S., arXiv:0902.0069 or Stanley, vol. 2, Exercise 5.59
- (Almost trivial) generalization of Lagrange inversion formula
- Comes in analytic-function and formal-power-series versions
- Recall Lagrange inversion: If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ with $a_1 \neq 0$ (as either analytic function or formal power series), then

$$f^{-1}(y) \;=\; \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] \left(\frac{\zeta}{f(\zeta)} \right)^m$$

  where $[\zeta^n] g(\zeta)$ denotes the coefficient of ζ^n in the power series $g(\zeta)$. More generally, if $h(x) = \sum_{n=0}^{\infty} b_n x^n$, we have

$$h(f^{-1}(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] \, h'(\zeta) \left(\frac{\zeta}{f(\zeta)} \right)^m$$

- Rewrite this in terms of $g(x) = x/f(x)$: then $f(x) = y$ becomes $x = g(x)y$, and its solution $x = \varphi(y) = f^{-1}(y)$ is given by the power series

$$\varphi(y) \;=\; \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] g(\zeta)^m$$

  and

$$h(\varphi(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] h'(\zeta) g(\zeta)^m$$

The explicit implicit function formula, continued

- Generalize $x = g(x)y$ to $x = G(x, y)$, where
  - $G(0,0) = 0$ and $|(\partial G/\partial x)(0,0)| < 1$ (analytic-function version)
  - $G(0,0) = 0$ and $(\partial G/\partial x)(0,0) = 0$ (formal-power-series version)
- Then there is a unique $\varphi(y)$ with zero constant term satisfying $\varphi(y) = G(\varphi(y), y)$, and it is given by

  $$\varphi(y) \;=\; \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}]G(\zeta, y)^m$$

  More generally, for any $H(x, y)$ we have

  $$H(\varphi(y), y) \;=\; H(0, y) \,+\, \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}] \frac{\partial H(\zeta, y)}{\partial \zeta}\, G(\zeta, y)^m$$

- Proof imitates standard proof of the Lagrange inversion formula: the variables y simply "go for the ride".
- Alternate interpretation: Solving fixed-point problem for the family of maps $x \mapsto G(x, y)$ parametrized by y. Variables y again "go for the ride".

Application to leading root of $f(x,y)$

- Start from a formal power series $f(x,y) = \sum_{n=0}^\infty a_n(y)\, x^n$ satisfying properties (a)–(c) above.
- Write out $f(-\xi_0(y), y) = 0$ and add $\xi_0(y)$ to both sides:

$$\xi_0(y) \;=\; a_0(y) \;-\; [a_1(y) - 1]\xi_0(y) \;+\; \sum_{n=2}^\infty a_n(y)\, (-\xi_0(y))^n$$

- Insert $\xi_0(y) = 1 + \varphi(y)$ where $\varphi(y)$ has zero constant term. Then $\varphi(y) = G(\varphi(y), y)$ where

$$G(z,y) \;=\; \sum_{n=0}^\infty (-1)^n\, \widehat{a}_n(y)\, (1+z)^n$$

  and

$$\widehat{a}_n(y) \;=\; \begin{cases} a_n(y) - 1 & \text{for } n = 0,1 \\ a_n(y) & \text{for } n \ge 2 \end{cases}$$

  And $\varphi(y)$ is the *unique* formal power series with zero constant term satisfying this fixed-point equation.

- Since this G satisfies $G(0,0) = 0$ and $(\partial G/\partial z)(0,0) = 0$ [indeed it satisfies the stronger condition $G(z,0) = 0$], we can apply the explicit implicit function formula to obtain an explicit formula for $\xi_0(y)$:

$$\xi_0(y) \;=\; 1 \;+\; \sum_{m=1}^\infty \frac{1}{m}\, [\zeta^{m-1}] \left(\sum_{n=0}^\infty (-1)^n\, \widehat{a}_n(y)\, (1+\zeta)^n \right)^m$$

  More generally, for any formal power series $H(z,y)$, we have

$$\begin{aligned} & H(\xi_0(y) - 1, y) \\ & \;=\; H(0,y) \;+\; \sum_{m=1}^\infty \frac{1}{m}\, [\zeta^{m-1}] \frac{\partial H(\zeta, y)}{\partial \zeta} \left(\sum_{n=0}^\infty (-1)^n\, \widehat{a}_n(y)\, (1+\zeta)^n \right)^m \end{aligned}$$

Application to leading root of $f(x,y)$, continued

- In particular, by taking $H(z,y) = (1+z)^\beta$ we can obtain an explicit formula for an arbitrary power of $\xi_0(y)$:

$$\xi_0(y)^\beta \;=\; 1 + \sum_{m=1}^{\infty} \frac{\beta}{m} \sum_{n_1,\ldots,n_m \ge 0} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} (-1)^{n_i} \widehat{a}_{n_i}(y)$$

- Important special case: $a_0(y) = a_1(y) = 1$ and $a_n(y) = \alpha_n\, y^{\lambda_n}$ $(n \ge 2)$ where $\lambda_n \ge 1$ and $\lim\limits_{n\to\infty} \lambda_n = \infty$. Then

$$[y^N] \frac{\xi_0(y)^\beta - 1}{\beta} \;=\; \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{n_1,\,\ldots,\,n_m \ge 2 \\ \sum\limits_{i=1}^{m} \lambda_{n_i} = N}} (-1)^{\sum n_i} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} \alpha_{n_i}$$

- Can this formula be used for proofs of nonnegativity???
- *Empirically* I know that the RHS is ≥ 0 when $\lambda_n = n(n-1)/2$:
  - For $\beta \ge -2$ with $\alpha_n = 1$ (partial theta function)
  - For $\beta \ge -1$ with $\alpha_n = 1/n!$ (deformed exponential function)
  - For $\beta \ge -1$ with $\alpha_n = (1-q)^n/(q;q)_n$ and $q > -1$
- And I can *prove* this (by a *different* method!) for the partial theta function (but not yet for the others).
- **How can we see these facts from this formula???**
  [open combinatorial problem]

Some positivity properties of formal power series

- Consider formal power series with real coefficients

$$f(y) \;=\; 1 + \sum_{m=1}^{\infty} a_m \, y^m$$

- For $\alpha \in \mathbb{R}$, define the class $\mathcal{S}_\alpha$ to consist of those f for which

$$\frac{f(y)^\alpha - 1}{\alpha} \;=\; \sum_{m=1}^{\infty} b_m(\alpha) \, y^m$$

  has all nonnegative coefficients (with a suitable limit when $\alpha = 0$).

- In other words:
  - For $\alpha > 0$ (resp. $\alpha = 0$), the class $\mathcal{S}_\alpha$ consists of those f for which f^α (resp. $\log f$) has all nonnegative coefficients.
  - For $\alpha < 0$, the class $\mathcal{S}_\alpha$ consists of those f for which f^α has all *nonpositive* coefficients after the constant term 1.

- Containment relations among the classes $\mathcal{S}_\alpha$ are given by the following fairly easy result:

  Proposition (Scott–A.D.S., unpublished):
  Let $\alpha, \beta \in \mathbb{R}$. Then $\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ if and only if either

  (a) $\alpha \le 0$ and $\beta \ge \alpha$, or
  (b) $\alpha > 0$ and $\beta \in \{\alpha, 2\alpha, 3\alpha, \ldots\}$.

  Moreover, the containment is strict whenever $\alpha \neq \beta$.

Application to deformed exponential function F

As shown earlier, it seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}
\xi_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\
& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
& + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\
& + \ldots + \text{terms through order } y^{899}
\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}
\xi_0(y)^{-1} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
& - \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\
& - \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\
& - \ldots - \text{terms through order } y^{899}
\end{aligned}$$

But I have no proof of either of these conjectures!!!

- Note that $\xi_0(y)$ is analytic on $0 \le y < 1$ and diverges as $y \uparrow 1$ like $1/[e(1-y)]$.
- It follows that $\xi_0(y) \notin \mathcal{S}_\alpha$ for $\alpha < -1$.

Application to partial theta function Θ_0

It seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}\xi_0(y) &= 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ &\qquad +948y^9 + 2610y^{10} + \ldots + \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}\xi_0(y)^{-1} &= 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ &\qquad -178y^9 - 490y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-2}$:

$$\begin{aligned}\xi_0(y)^{-2} &= 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ &\qquad -138y^9 - 386y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

Here I *do* have a proof of these properties (see below).

- Note that
$$\frac{\xi_0(y)^\alpha - 1}{\alpha} = y + \frac{\alpha+3}{2}y^2 + \frac{(\alpha+2)(\alpha+7)}{6}y^3 + O(y^4)$$
- So $\xi_0(y) \notin \mathcal{S}_\alpha$ for $\alpha < -2$.

Application to $\widetilde{R}(x,y,q) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(1+q)\,\cdots\,(1+q+\ldots+q^{n-1})}$

- Can use explicit implicit function formula to prove that

$$\xi_0(y;q) \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)}\, y^n$$

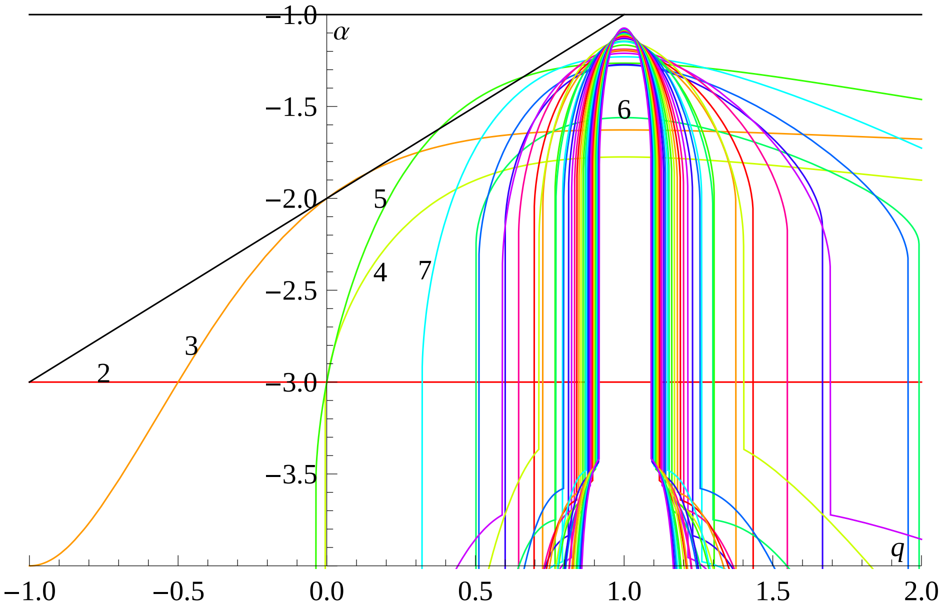
  where

$$Q_n(q) \;=\; \prod_{k=2}^{\infty} (1+q+\ldots+q^{k-1})^{\lfloor n/\binom{k}{2} \rfloor}$$

  and $P_n(q)$ is a self-inversive polynomial in q with integer coefficients.

- *Empirically* $P_n(q)$ has *two* interesting positivity properties:
  - (a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]\, P_5(q) = 0$.
  - (b) $P_n(q) > 0$ for $q > -1$.

- *Empirically* $\xi_0(y;q) \in \mathcal{S}_{-1}$ for all $q > -1$:



- **Can any of this be proven for $q \neq 0$?**

The deformed binomial series

Here is an even simpler family that interpolates between the partial theta function Θ_0 and the deformed exponential function F:

- Start from the Taylor series for the binomial $f(x) = (1-\mu x)^{-1/\mu}$
  [it is convenient to parametrize it in this way]
  and introduce factors $y^{n(n-1)/2}$ as usual:

$$
\begin{aligned}
F_\mu(x,y) &= \sum_{n=0}^{\infty} (-\mu)^n \binom{-1/\mu}{n} x^n \, y^{n(n-1)/2} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{j=0}^{n-1} (1+j\mu) \right) x^n \, y^{n(n-1)/2}
\end{aligned}
$$

- We call $F_\mu(x,y)$ the *deformed binomial function*.
- For $\mu = 0$ it reduces to the deformed exponential function.
- For $\mu = 1$ it reduces to the partial theta function.
- For $\mu = -1/N$ ($N = 1, 2, 3, \ldots$) it is a polynomial of degree N that is the "y-deformation" of the binomial $(1 + x/N)^N$

The deformed binomial series, continued

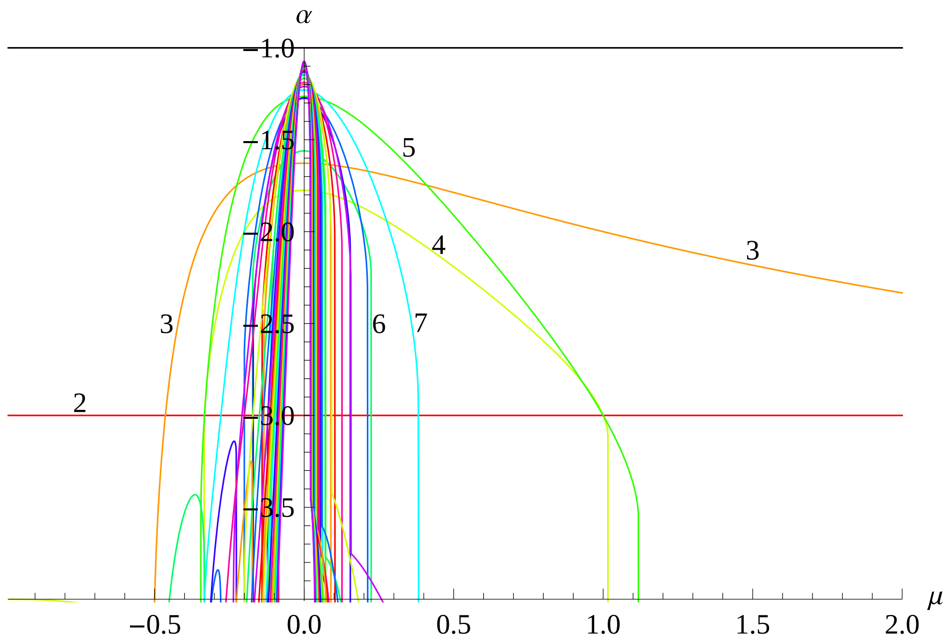
- Can use explicit implicit function formula to prove that

$$\xi_0(y;\mu) \;=\; 1 + \sum_{n=1}^{\infty} \frac{P_n(\mu)}{d_n} y^n$$

  where $P_n(\mu)$ is a polynomial of degree n with integer coefficients and d_n are explicit integers.

- *Empirically* $P_n(\mu)$ has *two* interesting positivity properties:
  (a) $P_n(\mu)$ has all strictly positive coefficients.
  (b) $P_n(\mu) > 0$ for $\mu > -1$.

- *Empirically* $\xi_0(y;\mu) \in \mathcal{S}_{-1}$ for all $\mu > -1$:



- **Can any of this be proven for $\mu \neq 1$?**

The deformed hypergeometric series

- Exponential (${}_0F_0$) and binomial (${}_1F_0$) are simplest cases of the hypergeometric series ${}_pF_q$.
- Can apply "y-deformation" process to ${}_pF_q$:

$$
{}_pF_q^*\left(\begin{matrix}\mu_1, \ldots, \mu_p \\ \nu_1, \ldots, \nu_q\end{matrix} \,\middle|\, x, y\right) \;=\; \sum_{n=0}^{\infty} \frac{(1;\mu_1)^{\overline{n}} \cdots (1;\mu_p)^{\overline{n}}}{(1;\nu_1)^{\overline{n}} \cdots (1;\nu_q)^{\overline{n}}} \, \frac{x^n}{n!} \, y^{n(n-1)/2}
$$

where

$$
(1;\mu)^{\overline{n}} \;=\; \prod_{j=0}^{n-1} (1 + j\mu)
$$

- Note that setting $\mu_p = 0$ reduces ${}_pF_q^*$ to ${}_{p-1}F_q^*$ (and likewise for ν_q).
- *Empirically* the two positivity properties for the deformed binomial appear to extend to ${}_2F_0^*$ (in the two variables μ_1, μ_2).
- I expect that this will generalize to all ${}_pF_0^*$.
- But the cases ${}_pF_q^*$ with $q \geq 1$ are different, and I do not yet know the complete pattern of behavior.

Identities for the partial theta function

- Use standard notation for q-shifted factorials:

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{for } |q|<1$$

- A pair of identities for the partial theta function:

$$\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} = (y;y)_\infty\,(-x;y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n\,(-x;y)_n}$$

$$\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} = (-x;y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\,(-x;y)_n}$$

as formal power series and as analytic functions on $(x,y) \in \mathbb{C} \times \mathbb{D}$

- Rewrite these as

$$\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} = (y;y)_\infty\,(-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\,(-xy;y)_{n-1}}\right]$$

$$\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} = (-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\,(-xy;y)_{n-1}}\right]$$

- The first identity goes back to Heine (1847).
- The second identity can be found in Andrews and Warnaar (2007) but is probably much older.

Proof that $\xi_0 \in \mathcal{S}_1$ for the partial theta function

- Let's say we use the first identity:

$$\Theta_0(x,y) \;=\; (y;y)_\infty \, (-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n \, (-xy;y)_{n-1}} \right]$$

- So $\Theta_0(x,y) = 0$ is equivalent to "brackets $= 0$".
- Insert $x = -\xi_0(y)$ and bring $\xi_0(y)$ to the LHS:

$$\xi_0(y) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n} (1-y^j) \prod\limits_{j=1}^{n-1} [1 - y^j \xi_0(y)]}$$

- This formula can be used iteratively to determine $\xi_0(y)$, and in particular to prove the strict positivity of its coefficients:
- Define the map $\mathcal{F}\colon\ \mathbb{Z}[[y]] \to \mathbb{Z}[[y]]$ by

$$(\mathcal{F}\xi)(y) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n} (1-y^j) \prod\limits_{j=1}^{n-1} [1 - y^j \xi(y)]}$$

- Define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{Z}[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$.
- Then $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \ldots \preceq \xi_0$ and $\xi_0^{(k)}(y) \;=\; \xi_0(y) \,+\, O(y^{3k+1})$.
- In particular, $\lim\limits_{k\to\infty} \xi_0^{(k)}(y) = \xi_0(y)$, and $\xi_0(y)$ has strictly positive coefficients.
- Thomas Prellberg has a combinatorial interpretation of $\xi_0(y)$ and $\xi_0^{(k)}(y)$.
- Proofs of $\xi_0 \in \mathcal{S}_{-1}$ and $\xi_0 \in \mathcal{S}_{-2}$ use second identity in a similar way.

A conjectured big picture

I conjecture that there are *three different* things going on here:

- **Positivity properties for the leading root $\boldsymbol{\xi_0(y)}$:**
  - $\xi_0(y)$ in various classes $\mathcal{S}_\beta$ for a fairly large class of series $f(x,y) = \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$
  - Appears to include deformed hypergeometric ${}_pF_0^*$, Rogers–Ramanujan $\widetilde{R}(x,y,q)$, probably others
  - Find sufficient conditions on $\{\alpha_n\}_{n=0}^{\infty}$??

- **Positivity properties for the higher roots $\boldsymbol{\xi_k(y)}$:**
  - *Some* positivity for partial theta function and perhaps others (needs further investigation)
  - Positivity of *all* $\xi_k(y)$ *only* for deformed exponential??

- **Positivity properties for ratios $\boldsymbol{\xi_k(y)/\xi_{k+1}(y)}$:**
  - Holds for some unknown class of series $f(x,y)$
  - Even for polynomials, class is unknown (cf. Calogero–Moser): roots should be "not too unevenly spaced"
  - Class appears to include at least deformed exponential
  - Needs much further investigation

Summary of open questions

- All the Big Conjectures concerning $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$.

- For a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n\, x^n\, y^{n(n-1)/2}$$

  with $\alpha_0 = \alpha_1 = 1$, find simple *sufficient* conditions to have $\xi_0(y) \succeq 0$ or more generally $\xi_0(y) \in \mathcal{S}_\beta$.

- In particular, want to handle $\alpha_n = 1/n!$ or $\alpha_n = (1-q)^n/(q;q)_n$ or $\alpha_n = (-\mu)^n \binom{-1/\mu}{n}$ or hypergeometric generalizations.

- Can this be done using explicit implicit function formula?
  (open combinatorial problem)

- Understand positivity properties for higher roots $x_k(y)$ and ratios of roots $x_k(y)/x_{k+1}(y)$.