Rational points on curves and quadratic Chabauty

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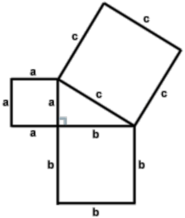
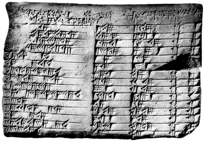
Finding rational points on curves



Pythagorean Theorem: $a^2 + b^2 = c^2$

Pythagoras
c. 570 - 495 BCE
MacTutor History of Mathematics

- We can find triples of integers (x, y, z) with $x^2 + y^2 = z^2$: for example, $(3, 4, 5), (5, 12, 13), (7, 24, 25), \ldots$
- This was known to the Babylonians



Plimpton 322 tablet,
c. 1800 BC

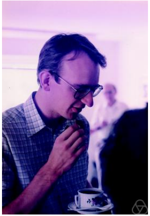
- In more modern language, we'd say that the triple of integers (x, y, z) gives us a *rational point* $\left(\frac{x}{z}, \frac{y}{z}\right)$ on the curve $X^2 + Y^2 = 1$, where $X := \frac{x}{z}, Y := \frac{y}{z}$

Rational points by genus

An invariant known as the *genus* of the curve tells us quite a bit about the arithmetic of the curve.

- Genus 0 (e.g., $x^2 + y^2 = 1$): rational points are understood by the Hasse principle
- Genus 1 (e.g., elliptic curves $y^2 = x^3 + 17$): rational points are finitely generated (but complicated)

For curves X of genus $g \geqslant 2$ (e.g., hyperelliptic curves, $y^2 = x^{2g+1} + c_{2g}x^{2g} + \cdots c_0$), it turns out that $X(\mathbf{Q})$ is finite:



Gerd Faltings
MFO

Theorem (Faltings, 1983)

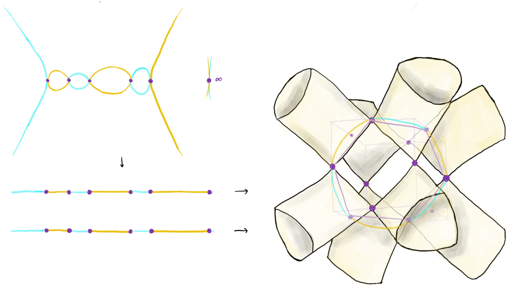
Let X be a curve of genus at least 2. Then $X(\mathbf{Q})$ is finite.

Note: Faltings' theorem is **not** constructive!

Motivating problem (Explicit Faltings): Given a curve $X/\mathbf{Q}$ with $g \geqslant 2$, compute $X(\mathbf{Q})$.

Working with higher genus curves

- For curves $X/\mathbf{Q}$ of genus at least 2, $X(\mathbf{Q})$ is just a set, so to study rational points, it helps to associate to X other objects that have more structure.
- Fix a basepoint $b \in X(\mathbf{Q})$. Embed X into its *Jacobian* J via the Abel-Jacobi map $\iota : X \hookrightarrow J$, sending $P \mapsto [(P) - (b)]$. The Mordell–Weil theorem tells us that $J(\mathbf{Q}) \cong \mathbf{Z}^r \oplus T$.
- The rank r is an important (but difficult to compute) invariant.



A genus 2 curve and its Kummer surface
Sachi Hashimoto

Sample problem: A question about triangles

We say a *rational* triangle is one with sides of rational lengths.

Question

Does there exist a rational right triangle and a rational isosceles triangle that have the same perimeter and the same area?

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Does there exist a rational right triangle and a rational isosceles triangle that have the same perimeter and the same area?

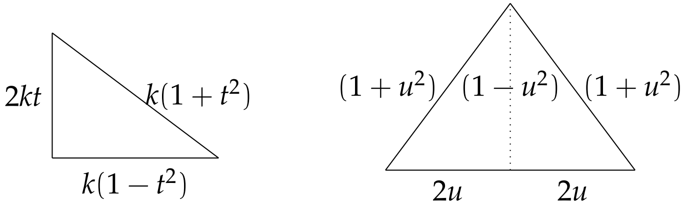
This feels like a very classical question but the answer is surprising – this was the result of work by Y. Hirakawa and H. Matsumura in 2018.

A question about triangles

Assume that there exists such a pair of triangles (rational right triangle, rational isosceles triangle). By rescaling both of the given triangles, we may assume their lengths are

$$(k(1+t^2), k(1-t^2), 2kt) \quad \text{and} \quad ((1+u^2), (1+u^2), 4u),$$

respectively, for some rational numbers $0 < t, u < 1, k > 0$.



A question about triangles

Given side lengths of

$$(k(1+t^2), k(1-t^2), 2kt) \quad \text{and} \quad ((1+u^2), (1+u^2), 4u),$$

by comparing perimeters and areas, we have

$$k + kt = 1 + 2u + u^2 \qquad \text{and} \qquad k^2 t(1-t^2) = 2u(1-u^2).$$

By a change of coordinates, this is equivalent to studying rational points on the genus 2 curve given by

$$X : y^2 = (3x^3 + 2x^2 - 6x + 4)^2 - 8x^6.$$

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So we consider the rational points on

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$$(0, \pm 4), (1, \pm 1), (2, \pm 8), (12/11, \pm 868/11^3), \infty^{\pm}$$

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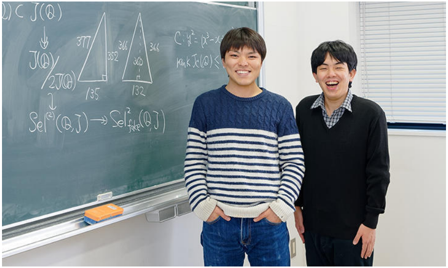
in $X(\mathbf{Q})$. We've found 10 points!
So we have provably determined $X(\mathbf{Q})$.

And $(12/11, 868/11^3)$ gives rise to a pair of triangles.

A question about triangles: answer

Theorem (Hirakawa–Matsumura, ’18)

Up to similitude, there exists a unique pair of a rational right triangle and a rational isosceles triangle that have the same perimeter and the same area. The unique pair consists of the right triangle with sides of lengths $(377, 135, 352)$ *and the isosceles triangle with sides of lengths* $(366, 366, 132)$.



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- ...and a bit of luck!

Example: Can we compute $X(\mathbf{Q})$?

Consider X:

$$-x^3y+2x^2y^2-xy^3-x^3z+x^2yz+xy^2z-2xyz^2+2y^2z^2+xz^3-3yz^3=0.$$

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The set $X(\mathbf{Q})$ contains 7 rational points (Galbraith):

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$$(1:0:0),(1:1:0),(0:3:2),(1:0:1).$$

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$$(1:0:0),(1:1:0),(0:3:2),(1:0:1).$$

Question: Is this set of points above precisely $X(\mathbf{Q})$?

Strategy for computing rational points on curves

Upshot: for *certain* curves X of genus at least 2, by associating other geometric objects to X, we can explicitly compute a slightly larger (but importantly, **finite**) set of points containing $X(\mathbf{Q})$, and then (hopefully) use this set to determine $X(\mathbf{Q})$.

- This story starts with the Chabauty–Coleman method.
- There are many variations on the Chabauty–Coleman method (by Bruin, Flynn, Siksek, Stoll, Wetherell,...) that have also tackled a number of interesting curves in higher rank.
- There is a vast generalization of this (*nonabelian Chabauty*, a program initiated by Kim) to understand rational points on curves with Jacobians of higher rank.

Chabauty's theorem

Theorem (Chabauty, '41)

Let X be a curve of genus $g \geqslant 2$ over $\mathbf{Q}$. Suppose the Mordell-Weil rank r of $J(\mathbf{Q})$ is less than g. Then $X(\mathbf{Q})$ is finite.

- Coleman (1985) made Chabauty's theorem effective by re-interpreting this result in terms of p-adic line integrals of regular 1-forms.
- In fact, by counting the number of zeros of such an integral, Coleman gave the bound

$$\#X(\mathbf{Q}) \leqslant \#X(\mathbf{F}_p) + 2g - 2.$$

Robert Coleman
MFO

The method of Chabauty–Coleman

Let $p > 2$ be a prime of good reduction for X. The map $H^0(J_{\mathbf{Q}_p}, \Omega^1) \longrightarrow H^0(X_{\mathbf{Q}_p}, \Omega^1)$ induced by ι is an isomorphism of $\mathbf{Q}_p$-vector spaces. Suppose ω_J restricts to ω.

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If $r < g$, there exists $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$ such that

$$\int_b^P \omega = 0$$

for all $P \in X(\mathbf{Q})$. Thus by studying the zeros of $\int \omega$, we can find a finite set of p-adic points containing the rational points of X.

Recap of the method (+bonus observations)

Given a curve $X/\mathbf{Q}$ of genus $g \geqslant 2$, embed it inside its *Jacobian* J and consider the rank r of $J(\mathbf{Q})$.

- If $r < g$, we can use the Chabauty–Coleman method to compute a regular 1-form whose p-adic (Coleman) integral vanishes on rational points.
- By studying the zeros of this integral, Coleman gave the bound
$$\#X(\mathbf{Q}) \leqslant \#X(\mathbf{F}_p) + 2g - 2.$$
- This bound can be sharp in practice, as in the triangle example ($g = 2, r = 1, p = 5$ gave $\#X(\mathbf{F}_p) = 8$ and thus $\#X(\mathbf{Q}) \leqslant 10$).
- Regardless, the Coleman integral cuts out a finite set of p-adic points; this set contains $X(\mathbf{Q})$ as a subset.
- Even when the bound is not sharp, we can often combine Chabauty–Coleman data at multiple primes (Mordell–Weil sieve) to extract $X(\mathbf{Q})$.

Computing rational points via Chabauty–Coleman

We have

$$X(\mathbf{Q}) \subset X(\mathbf{Q}_p)_1 := \left\{ z \in X(\mathbf{Q}_p) : \int_b^z \omega = 0 \right\}$$

for a p-adic line integral $\int_b^* \omega$, with $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$.

We would like to compute an annihilating differential ω and then calculate the finite set of p-adic points $X(\mathbf{Q}_p)_1$.

Example: Chabauty–Coleman with $g = 2, r = 1$

Suppose we have a genus 2 curve $X/\mathbf{Q}$ with $\operatorname{rk} J(\mathbf{Q}) = 1$ and $X(\mathbf{Q}) \neq \emptyset$. Fix a basepoint $b \in X(\mathbf{Q})$.

- We know $H^0(X_{\mathbf{Q}_p}, \Omega^1) = \langle \omega_0, \omega_1 \rangle$.
- Since $r = 1 < 2 = g$, we can compute $X(\mathbf{Q}_p)_1$ as the zero set of a p-adic integral.
- If we know one more point $P \in X(\mathbf{Q})$, we can compute the constants $A, B \in \mathbf{Q}_p$:

$$\int_b^P \omega_0 = A, \quad \int_b^P \omega_1 = B,$$

then solve the equation

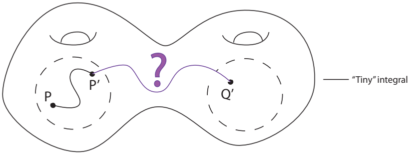
$$f(z) := \int_b^z (B\omega_0 - A\omega_1) = 0$$

for $z \in X(\mathbf{Q}_p)$.

- The set of such z is finite, and $X(\mathbf{Q})$ is contained in this set.

A word on p-adic integration

Coleman integrals are p-adic *line integrals*.



p-adic line integration is difficult – how do we construct the correct path?

- We can construct local ("tiny") integrals easily, but extending them to the entire space is challenging.
- Coleman's solution: *analytic continuation along Frobenius*, giving rise to a theory of p-adic line integration satisfying the usual nice properties: linearity, additivity, change of variables, fundamental theorem of calculus.
- Implementations in **SageMath** for hyperelliptic curves (B.–Bradshaw–Kedlaya) and in **Magma** (GitHub) for smooth curves (B.–Tuitman)

For which curves X do we want to compute $X(\mathbf{Q})$?

Sometimes curves are interesting for historic reasons.

Diophantus of Alexandria was a Greek mathematician who lived in the third century. In Problem 17 of book VI of the Arabic manuscript of *Arithmetica*, Diophantus poses the following problem:

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In other words, Diophantus asked if one can find positive rational x and y such that the equation

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is satisfied.

He gave the solution $x = 1/2, y = 9/16$. Are there any others?

Diophantus' curve

Removing the singularity of the curve at $(0,0)$, this amounts to determining the set of *all* rational points on the genus 2 curve Y with affine model

$$y^2 = x^6 + x^2 + 1.$$

This curve is interesting because it is the only higher genus curve considered in the 10 known books of the *Arithmetica*. Moreover, this curve is of interest because it lies just beyond the boundary of what is feasible using the Chabauty–Coleman method.

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Theorem (Wetherell, '97)

The set of rational points $Y(\mathbf{Q})$ is precisely

$$\left\{(0,\pm 1), \left(\pm\frac{1}{2}, \pm\frac{9}{8}\right), \infty^{\pm}\right\}.$$

From Diophantus to Wetherell

One special property of the genus 2 curve Y is that it has extra symmetries and is said to be *bielliptic*: indeed, these extra automorphisms of the curve result in a nice decomposition of its Jacobian into a product of two elliptic curves.

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Wetherell gave a solution to Diophantus' problem by considering a collection of covering curves

$$\{F_i\} \to Y$$

and applying the Chabauty–Coleman method on the covers F_i, from which the result about $Y(\mathbf{Q})$ follows.

Covering collections

A *cover* of a curve C is a surjective map from a curve F onto C.

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Idea: construct a cover F (or a collection of covering curves $\{F_i\}$) of C such that every rational point on C comes from a rational point on F (on one of the F_i, respectively). Then one can compute $C(\mathbf{Q})$ by computing $F(\mathbf{Q})$ (the sets $F_i(\mathbf{Q})$, respectively).

- Since the Jacobian of Y is isogenous to the product of two rank 1 elliptic curves, one has particularly nice covers of Y.
- Wetherell found genus 5 curves G_1 and G_2 covering Y.
- Taking the quotient of each G_i by certain automorphisms gives two genus 3 curves F_i, such that determining $F_1(\mathbf{Q})$ and $F_2(\mathbf{Q})$ would result in computing the set $Y(\mathbf{Q})$.
- The genus 3 curves F_1 and F_2 had Jacobians with rank 1 and 0, and Wetherell successfully carried out the Chabauty–Coleman method on F_1 and F_2 to deduce the theorem about $Y(\mathbf{Q})$.

For which curves X do we want to compute $X(\mathbf{Q})$?

There are a number of fundamental questions in number theory that come from moduli problems, in particular, understanding rational points on *modular curves*, e.g.:

Theorem (Mazur, 1977)

If $E/\mathbf{Q}$ is an elliptic curve, and $P \in E(\mathbf{Q})$ has finite order N, then $N \in \{1, \ldots, 10, 12\}$.

Idea: Find the rational points on the modular curve $X_1(N)$.

- Non-cuspidal points in $X_1(N)(\mathbf{Q})$ correspond to elliptic curves $E/\mathbf{Q}$ with a point $P \in E(\mathbf{Q})$ of order N.
- So Mazur's theorem is equivalent to the assertion that $X_1(N)(\mathbf{Q})$ consists only of cusps if $N = 11$ or $N \geqslant 13$.

Residual Galois representations

Let $E/\mathbf{Q}$ be an elliptic curve, ℓ a prime number.

- $G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on the ℓ-torsion points $E[\ell]$.
- Fixing a basis of $E[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^2$, get a Galois representation

$$\bar{\rho}_{E,\ell} : G_{\mathbf{Q}} \to \mathrm{Aut}(E[\ell]) \cong \mathbf{GL}_2(\mathbf{F}_\ell)$$

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Serre's uniformity problem: Does there exist an absolute constant ℓ_0 such that $\bar{\rho}_{E,\ell}$ is surjective for every non-CM elliptic curve $E/\mathbf{Q}$ and every prime $\ell > \ell_0$?

Folklore: $\ell_0 = 37$ should work.

Serre's Uniformity Problem

Idea: To show that $\bar{\rho}_{E,\ell}$ is surjective, show that $\mathrm{im}(\bar{\rho}_{E,\ell})$ is not contained in a maximal subgroup of $\mathbf{GL}_2(\mathbf{F}_\ell)$. These are

1. Borel subgroups
2. Exceptional subgroups
3. Normalizers of split Cartan subgroups
4. Normalizers of non-split Cartan subgroups

Idea: For a maximal $G \subset \mathbf{GL}_2(\mathbf{F}_\ell)$, there is a modular curve $X_G/\mathbf{Q}$ such that non-cuspidal points in $X_G(\mathbf{Q})$ correspond to elliptic curves $E/\mathbf{Q}$ with $\mathrm{im}(\bar{\rho}_{E,\ell}) \subset G$.

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2. Exceptional subgroups ✓(Serre)
3. Normalizers of split Cartan subgroups ✓(Bilu–Parent–Rebolledo)
4. Normalizers of non-split Cartan subgroups

Idea: For a maximal $G \subset \mathbf{GL}_2(\mathbf{F}_\ell)$, there is a modular curve $X_G/\mathbf{Q}$ such that non-cuspidal points in $X_G(\mathbf{Q})$ correspond to elliptic curves $E/\mathbf{Q}$ with $\operatorname{im}(\bar{\rho}_{E,\ell}) \subset G$.

Serre's Uniformity Problem

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The cursed modular curve

All normalizers of split Cartan $G \subset \mathbf{GL}_2(\mathbf{F}_\ell)$ are conjugate, so all corresponding $X_G = X(\ell)/G$ are isomorphic. Denote $X_{\mathrm{s}}(\ell) = X_G$.

Theorem (Bilu-Parent 2011, Bilu-Parent-Rebolledo 2013)
We have $X_{\mathrm{s}}(\ell)(\mathbf{Q}) = \{$*cusps, CM-points*$\}$ *for* $\ell \geqslant 11$, $\ell \neq 13$.

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What goes wrong at $\ell = 13$? Bilu-Parent-Rebolledo refer to $\ell = 13$ as the "cursed" level; crucial to their method is Mazur's method for integrality of non-cuspidal rational points, using the following:

$$\mathrm{Jac}(X_{\mathrm{s}}(\ell)) \sim \mathrm{Jac}(X_0^+(\ell^2)) \sim J_0(\ell) \times \mathrm{Jac}(X_{\mathrm{ns}}(\ell))$$

Curses of the cursed curve

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- Mazur's method applies whenever $J_0(\ell) \neq 0$, which is the case for $\ell = 11$ and $\ell \geqslant 17$.
- But for $\ell = 13$, we have $J_0(13) = 0$.
- Curse #1: We thus have $\mathrm{Jac}(X_{\mathrm{s}}(13)) \sim \mathrm{Jac}(X_{\mathrm{ns}}(13))$ and $\mathrm{Jac}(X_{\mathrm{s}}(13))$ is absolutely simple.

Curses of the cursed curve, continued

Curse #2: Baran found an explicit smooth plane quartic model and showed

$$X_{\mathrm{s}}(13) \simeq_{\mathbf{Q}} X_{\mathrm{ns}}(13),$$

its non-split analogue. (No modular explanation for this!)

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$$X : x^3y + x^3z - 2x^2y^2 - x^2yz + xy^3 - xy^2z + 2xyz^2 - xz^3 - 2y^2z^2 + 3yz^3 = 0.$$

Visualizations of the cursed curve
JB and Sachi Hashimoto

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Curse #3: $r = \operatorname{rk} J(\mathbf{Q}) \geqslant 3 = g$. ☹

Beyond Chabauty–Coleman

Do we have any hope of doing something like Chabauty–Coleman when $r \geqslant g$?

- Conjecturally, yes, via Kim's nonabelian Chabauty program.
- Instead of using the Jacobian of X and abelian integrals, use *nonabelian geometric objects* associated to X, which carry *iterated* Coleman integrals.
- These iterated integrals cut out Selmer varieties, which give a sequence of sets

  $$X(\mathbf{Q}) \subset \cdots \subset X(\mathbf{Q}_p)_n \subset X(\mathbf{Q}_p)_{n-1} \subset \cdots \subset X(\mathbf{Q}_p)_2 \subset X(\mathbf{Q}_p)_1$$

  where the depth n set $X(\mathbf{Q}_p)_n$ is given by equations in terms of n-fold iterated Coleman integrals

  $$\int_b^P \omega_n \cdots \omega_1.$$

- Note that $X(\mathbf{Q}_p)_1$ is the classical Chabauty–Coleman set.

Nonabelian Chabauty

Conjecture (Kim, '12)
For $n \gg 0$, the set $X(\mathbf{Q}_p)_n$ is finite.

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Questions:

- When can $X(\mathbf{Q}_p)_n$ be shown to be finite?
- For which classes of curves can nonabelian Chabauty be used to prove Faltings’ theorem?

Finiteness of $X(\mathbf{Q}_p)_n$

Theorem (Coates–Kim '10)

For $X/\mathbf{Q}$ with CM Jacobian, for $n \gg 0$, the set $X(\mathbf{Q}_p)_n$ is finite.

Theorem (Ellenberg–Hast '17)

Can extend the above to give a new proof of Faltings' theorem for curves $X/\mathbf{Q}$ that are solvable Galois covers of $\mathbf{P}^1$.

Theorem (B.–Dogra '16)

For $X/\mathbf{Q}$ with $g \geqslant 2$ and

$$r < g + \operatorname{rk} NS(J_{\mathbf{Q}}) - 1,$$

the set $X(\mathbf{Q}_p)_2$ is finite.

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So when can we explicitly compute $X(\mathbf{Q}_p)_2$? We call this *quadratic Chabauty*.

Quadratic Chabauty: $\mathbf{Q}$-points and p-adic heights

Want to use "quadratic Chabauty" to compute $X(\mathbf{Q}_p)_2$, a finite set of p-adic points that contains all rational points on X for certain curves that have $r = g$

- We know that $X(\mathbf{Q}_p)_2$ is finite when $r = g$ and $\operatorname{rk} NS(J) > 1$. The difficulty is in making this effective.
- The functions cutting out p-adic points can be expressed in terms of p-adic heights pairings; the key is to move from linear relations (as in Chabauty–Coleman) to bilinear relations.
- These p-adic heights have a natural interpretation in terms of p-adic differential equations, with relevant constants computed in terms of known rational points.

Dictionary between classical and quadratic Chabauty

technique	classical Chabauty	quadratic Chabauty
hypotheses	$r < g$	$r = g$ and $\mathrm{rk}\, NS(J_{\mathbf{Q}}) \geqslant 2$
geometry	Jacobian	Selmer variety
p-adic analysis	line integrals	iterated path integrals
algebra	linear algebra	bilinear algebra (heights)

Quadratic Chabauty (roughly)

Given a global p-adic height h, we study it on rational points:

$$\underbrace{h}_{\substack{\text{bilinear form, rewrite in terms}\\ \text{of locally analytic function}\\ \text{using known rational points}}} = \underbrace{h_p}_{\substack{\text{locally analytic function}\\ \text{via } p\text{-adic differential equation}}} + \underbrace{\sum_{v \neq p} h_v}_{\substack{\text{takes on finite}\\ \text{number of values}\\ \text{on rational points}\\ \text{(best case: all trivial)}}}$$

Note: to determine the local height h_p, need to compute Frobenius structure on the relevant p-adic differential equation.

- This amounts to computing the action of Frobenius on p-adic cohomology.

Constructing quadratic Chabauty functions

- Construct a quadratic Chabauty function by associating to points of X certain p-adic Galois representations, and then take Nekovář p-adic heights.
- Idea is to construct a representation $A(x)$ for every $x \in X(\mathbf{Q})$. Depends on a choice of "nice" correspondence Z on X. Such a correspondence exists when $\operatorname{rk} NS(J) > 1$.
- Restrict to case of X with everywhere potential good reduction, then for all $v \neq p$, local heights $h_v(A(x))$ are trivial.
- Compute p-adic height of $A(x)$ via p-adic Hodge theory.

Quadratic Chabauty for rational points

- Using Nekovář's p-adic height h, there is a local decomposition

$$h(A(x)) = h_p(A(x)) + \sum_{v \neq p} h_v(A(x))$$

where

1. $x \mapsto h_p(A(x))$ extends to a locally analytic function $\theta : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ by Nekovář's construction and
2. For $v \neq p$ the local heights $h_v(A(x))$ are trivial since by assumption, all primes $v \neq p$ are of potential good reduction

This gives a quadratic Chabauty function whose pairing is h and whose endomorphism is induced by Z.

Quadratic Chabauty

Suppose $X/\mathbf{Q}$ satisfies

- $r = g$,
- $\operatorname{rk} NS(J_{\mathbf{Q}}) > 1$,
- p-adic closure $\overline{J(\mathbf{Q})}$ has finite index in $J(\mathbf{Q}_p)$,
- X has everywhere potential good reduction
- and that we know enough rational points $P_i \in X(\mathbf{Q})$.

If we can solve the following problems, we have an algorithm for computing a finite subset of $X(\mathbf{Q}_p)$ containing $X(\mathbf{Q})$:

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If we can solve the following problems, we have an algorithm for computing a finite subset of $X(\mathbf{Q}_p)$ containing $X(\mathbf{Q})$:

1. Expand the function $x \mapsto h_p(A(x))$ into a p-adic power series on every residue disk.
2. Evaluate $h(A(P_i))$ for the known rational points $P_i \in X(\mathbf{Q})$.

Note that since we are assuming we have everywhere potentially good reduction, we have

$$h(A(x)) = h_p(A(x)),$$

i.e., the second problem is subsumed by the first.

High-level strategy: QC for the cursed curve

Practical matters:

- Show that $X_s(13)$ has $r = 3$.
- Make a small change of coordinates to work with the following curve X:

  $$Q(x, y, z) = y^4 + 5x^4 - 6x^2y^2 + 6x^3z + 26x^2yz + 10xy^2z - 10y^3z - 32x^2z^2 - 40xyz^2 + 24y^2z^2 + 32xz^3 - 16yz^3 = 0$$

  so that we have enough (5 of the known) rational points in each of two affine patches.
- Since $\mathrm{rk}\, NS(J_{\mathbf{Q}}) = 3$, we have two independent nontrivial nice correspondences Z_1, Z_2 on X; we compute equations for 17-adic heights h^{Z_1}, h^{Z_2} on X
- Check the simultaneous solutions of the above two equations...are they precisely on the 7 known rational points?!

Rational points on the cursed curve

Theorem (B.–Dogra–Müller–Tuitman–Vonk)

We have $|X_s(13)(\mathbf{Q})| = 7$.

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By the work of Baran, we know $X_{\mathrm{s}}(13)$ is isomorphic to $X_{\mathrm{ns}}(13)$ over $\mathbf{Q}$, so we also get (for free) that $|X_{\mathrm{ns}}(13)(\mathbf{Q})| = 7$.

Does the curse continue?

Consider the following smooth plane quartic, studied by Banwait–Cremona:

$$X_{S_4}(13) : 4x^3y - 3x^2y^2 + 3xy^3 - x^3z + 16x^2yz - 11xy^2z + 5y^3z + 3x^2z^2 + 9xyz^2 + y^2z^2 + xz^3 + 2yz^3 = 0.$$

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Theorem (B.–Dogra–Müller–Tuitman–Vonk)

We have $|X_{S_4}(13)(\mathbf{Q})| = 4$.