A Lie group analog for the monster Lie algebra

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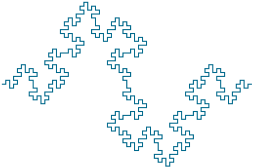

Work in progress with Elizabeth Jurisich and Scott H. Murray.

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We begin in the realm of infinite dimensional Lie algebras

Kac–Moody algebras are infinite dimensional Lie algebras which are the most natural generalization to infinite dimensions of finite dimensional simple Lie algebras.

A generalization of Kac-Moody algebras proposed by Borcherds, called *Borcherds algebras*, or *generalized Kac-Moody algebras*, can be constructed, in many interesting cases, from 'physical' subspaces of vertex operator algebras in the context of conformal field theory.

Our primary example of a Borcherds algebra is the *monster Lie algebra* $\mathfrak{m}$, which admits an action of the Monster finite simple group $\mathbf{M}$. This is the largest of the 26 sporadic simple groups.

The monster Lie algebra $\mathfrak{m}$ was constructed by Borcherds as part of his program to solve the Conway-Norton conjecture about the representation theory of the Monster finite simple group $\mathbf{M}$.

Some background

Borcherds' construction of the monster Lie algebra $\mathfrak{m}$ is based on Frenkel, Lepowsky and Meurman's moonshine module $V^\natural$. This is a vertex operator algebra whose automorphism group is $\mathbf{M}$.

In addition to its profound mathematical importance, the monster Lie algebra $\mathfrak{m}$ has been realized as symmetries in heterotic string theory (noted by Harvey and Moore and more recently by Paquette, Persson and Volpato).

Since Borcherds algebras describe physical symmetries, it is important to associate to them a group which is an analog of a Lie group.

In particular this would give a global group structure which ideally can be interpreted as a group of transformations.

Today we will outline approaches to some of the first constructions of Lie group analogs for infinite dimensional Borcherds algebras.

THE MONSTER LIE ALGEBRA

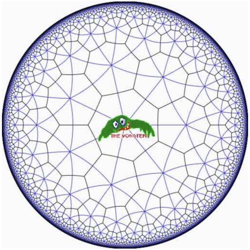
M, the Monster finite simple group with order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 = 8 \times 10^{53}$$

$\mathfrak{m}$, the monster Lie algebra of Borcherds, a quotient of the 'physical space' P_1 of the vertex operator algebra $V = V^\natural \otimes V_{1,1}$

$V^\natural$, the moonshine module of Frenkel, Lepowsky and Meurman, a graded **M**-module with $\mathrm{Aut}\,(V^\natural) = \mathbf{M}$

$V_{1,1}$, a vertex operator algebra for the even unimodular 2-dim Lorentzian lattice $II_{1,1}$, a trivial **M** module



GENERATORS AND RELATIONS FOR $\mathfrak{m}$

The monster Lie algebra also has a realization in terms of generators and relations $\mathfrak{m} \cong \mathfrak{g}(A)/\mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}(A)$. The data for the Lie algebra relations is encoded in the infinite generalized Cartan matrix A:

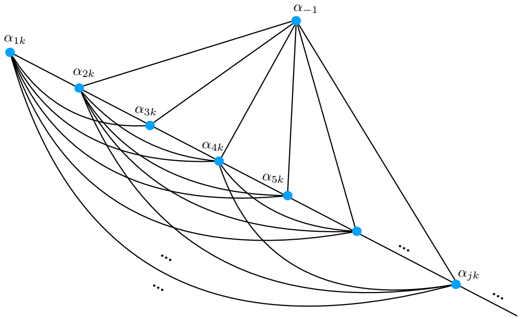
$$A = \begin{array}{cc}
 & \begin{array}{c|ccc|ccc|c} c(-1) & & c(1) & & & c(2) & & \end{array} \\
\begin{array}{c} c(-1) \\ \\ c(1) \\ \\ \\ c(2) \\ \\ \\ \end{array} & \left(\begin{array}{c|ccc|ccc|c}
2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\
\hline
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\hline
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\hline
\vdots & & \vdots & & & \vdots & &
\end{array}\right)
\end{array}$$

and the numbers $c(j)$ are coefficients of q in the modular function $J(q) = j(q) - 744 =$

$$\sum_{i \geq -1} c(j) q^j = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

where $c(-1) = 1$, $c(0) = 0$, $c(1) = 196884, \ldots$.

DYNKIN DIAGRAM OF $\mathfrak{m}$



Dynkin diagram for the Monster Lie algebra $\mathfrak{m}$ with simple roots α_{-1} and α_{jk}. This is a complete graph on a countable set, missing only one egde, namely the edge between α_{-1} and α_{1k}. The multiplicities on the edges are omitted. Edges between α_{-1} and α_{jk} have multiplicity $|1-j|$ and edges between α_{jk} and $\alpha_{j'k'}$ have multiplicity $|-(j+j')|$. Each node represents one block of the matrix A.

Main question

There are many methods for associating Lie group analogs to Kac–Moody algebras.

Until recently, group constructions associated to Borcherds algebras remained elusive.

We consider the problem of associating an analog of a Lie group to the monster Lie algebra $\mathfrak{m}$.

We seek a group of automorphisms of $\mathfrak{m}$ if possible.

We will discuss why the methods for associating Lie groups to finite dimensional Lie algebras break down.

The exponential map and the adjoint representation

Let $\mathfrak{g}$ be a simple Lie algebra and let G be the corresponding simple Lie group. We have the formal exponential

$$\begin{aligned} \exp\ : \mathfrak{g} &\to G \\ X &\mapsto \sum_{k=0}^{\infty} \frac{1}{k!} X^k \end{aligned}$$

When $\mathfrak{g}$ is finite dimensional, this is a finite (terminating) series when applied to the endomorphism $\mathrm{ad}\,(x) : y \mapsto [x, y]$. The map

$$\begin{aligned} \mathrm{ad}\ : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}\,(x) \end{aligned}$$

is called the *adjoint representation*. Hence

$$(\exp\,(\mathrm{ad}\,(x))(y) = y + [x, y] + \tfrac{1}{2!}[x, [x, y]] + \cdots$$

is finite and thus $\exp\,(\mathrm{ad}\,(x))$ is a well-defined element of $\mathrm{Aut}\,(\mathfrak{g})$.

Lie groups generated by elements $\exp(\mathrm{ad}(x))$

Let $\mathfrak{g}$ be a simple Lie algebra. We may associate a Lie group to $\mathfrak{g}$ of the form

$$G = \langle \exp(s \cdot \mathrm{ad}(e_\alpha)), \exp(t \cdot \mathrm{ad}(f_\alpha)) \mid \alpha \in \Delta,\ s, t \in \mathbb{C} \rangle$$

where $\{e_\alpha, f_\alpha, h_\alpha \mid \alpha \in \Delta, h_\alpha \in \mathfrak{h}\}$ is a 'Chevalley' basis for $\mathfrak{g}$ and $s, t \in \mathbb{C}$. The elements

$$\exp(s \cdot \mathrm{ad}(e_\alpha)) = I + s \cdot \mathrm{ad}(e_\alpha) + \frac{s^2}{2} \cdot (\mathrm{ad}(e_\alpha))^2 + \dots$$

$$\exp(t \cdot \mathrm{ad}(f_\alpha)) = I + t \cdot \mathrm{ad}(f_\alpha) + \frac{t^2}{2} \cdot (\mathrm{ad}(f_\alpha))^2 + \dots$$

are finite sums and hence well defined elements of $\mathrm{Aut}(\mathfrak{g})$. We obtain a representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. The *Chevalley group of adjoint type* is defined as

$$\mathrm{Ad}(G) \leq \mathrm{Aut}(\mathfrak{g}).$$

Example: the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is the 3-dimensional Lie algebra of 2×2 matrices of trace 0 over $\mathbb{C}$ generated by symbols

$$h,\ e,\ f$$

such that

$$[h,h] = 0,\ [e,f] = h,\ [h,e] = 2e \text{ and } [h,f] = -2f.$$

The adjoint group $\mathrm{Ad}\,(G) \leq \mathrm{Aut}\,(\mathfrak{g})$ generated by $\exp\,(s \cdot \mathrm{ad}\ e)$ and $\exp\,(t \cdot \mathrm{ad}\ f)$, for $s, t \in \mathbb{C}$, is isomorphic to (a 3-dimensional representation of) $\mathrm{PSL}_2(\mathbb{C})$.

'exp' AND 'ad' FOR INFINITE DIMENSIONAL LIE ALGEBRAS

A similar method works for constructing adjoint groups for (infinite dimensional) Kac–Moody algebras.

The adjoint representation $\mathrm{ad}(\mathfrak{g})$ of a Kac–Moody algebra $\mathfrak{g}$ is *locally nilpotent* on simple root vectors e_i and f_i for $i = 1, \ldots, \ell$. That is

$$(\mathrm{ad}(e_i))^{n_i} = 0, \quad (\mathrm{ad}(f_i))^{m_i} = 0$$

for some $n_i, m_i \gg 0$. Thus $\exp(\mathrm{ad}(e_i))$ and $\exp(\mathrm{ad}(f_i))$ are finite sums and hence well defined elements of $\mathrm{Aut}(\mathfrak{g})$. We may consider the group generated by

$$\exp(s \cdot \mathrm{ad}(e_i)) \text{ and } \exp(t \cdot \mathrm{ad}(f_i))$$

for $i = 1, \ldots, \ell$ and $s, t \in \mathbb{C}$.

Adjoint constructions of Kac–Moody groups are well known. There is also a notion of a 'simply connected' Chevalley group where the adjoint representation is replaced by a highest weight representation.

These approaches no longer work for Borcherds algebras.

The roots of a Lie algebra

To discuss the reasons why these methods break down for Borcherds algebras, we consider the following definition.

The *roots* of a Lie algebra are the eigenvalues of its adjoint representation.

The *simple roots* are a linearly independent subset of roots that generates the lattice of all roots.

A choice of eigenvectors for the simple roots are called *simple root vectors* and they generate the whole Lie algebra.

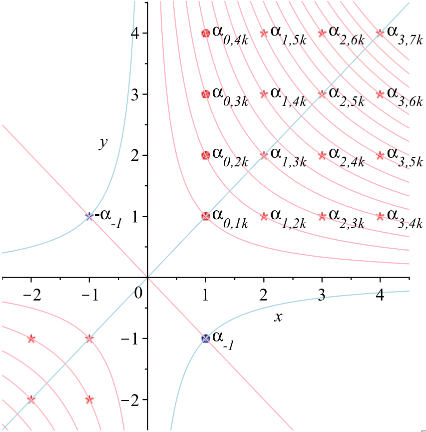
The roots of a Lie algebra can be realized geometrically in a real vector space.

A root α is called *real* if $\|\alpha\|^2 \geq 0$ and *imaginary* if $\|\alpha\|^2 < 0$.

The simple roots of a finite dimensional Lie algebra or Kac–Moody algebra are real roots, while Borcherds algebras may have imaginary simple roots.

Root lattice $II_{1,1}$ for $\mathfrak{m}$

We identify the root lattice of $\mathfrak{m}$ with the Lorentzian lattice $II_{1,1}$ which is $\mathbb{Z} \oplus \mathbb{Z}$ equipped with the bilinear form given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Simple roots are α_{-1} and $\alpha_{0,jk}$, $j \in \mathbb{N}$. Real roots are denoted by blue nodes on the blue hyperbola with squared norm 2, imaginary roots are denoted by red nodes on the red hyperbolas with squared norms ≤ -2.



‘$\exp \operatorname{ad}$’ FOR BORCHERDS ALGEBRAS

For the monster Lie algebra $\mathfrak{m}$, ad is not locally nilpotent on imaginary simple root vectors and thus

$$\exp(\operatorname{ad}(x)(y)) = y + [x, y] + \frac{1}{2!}[x, [x, y]] + \cdots$$

is generally an infinite sum and hence not a well-defined automorphism of $\mathfrak{m}$.

We need a different approach for $\mathfrak{m}$.

Lie groups defined by generators and relations

Steinberg gave a defining presentation for finite dimensional (adjoint and simply connected) Chevalley groups over commutative rings.

Tits gave generators and relations for Kac–Moody groups, generalizing the Steinberg presentation.

These presentations by Steinberg and Tits can be realized as groups of automorphisms of the underlying Lie algebras by identifying the generators with elements of the form $\exp \operatorname{ad}(x)$ for suitable x.

We may define a group for the monster Lie algebra $\mathfrak{m}$ in terms of generators and relations by mimicking the Steinberg and Tits presentations, but this will be an abstract group, not a group of automorphisms of $\mathfrak{m}$.

Structure of $\mathfrak{m}$

To discuss our approaches to these obstacles, we consider the following.

The monster Lie algebra $\mathfrak{m}$ has decomposition (Jurisich)

$$\mathfrak{m} = \mathfrak{u}_-^{im} \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}_+^{im}$$

where

$$\mathfrak{u}_+^{im} = L(S) \text{ and } \mathfrak{u}_-^{im} = L(S')$$

are free Lie algebras on countably many imaginary root vectors of $\mathfrak{m}$, indexed by sets S and S' respectively:

$$S = \cup_{j\in\mathbb{N}}\{(\mathrm{ad}\, e_{-1})^{\ell} e_{jk} \mid 0 \leq \ell < j, 1 \leq k \leq c(j)\}$$
$$= \left\{e_{jk},\ [e_{-1}, e_{jk}],\ [e_{-1}, [e_{-1}, e_{jk}]],\ [e_{-1}, [e_{-1}, [e_{-1}, e_{jk}]]], \ldots, (\mathrm{ad}\, e_{-1})^{j-1} e_{jk})\right\}_{j\in\mathbb{N},\ k\leq c(j)}$$
$$S' = \cup_{j\in\mathbb{N}}\{(\mathrm{ad}\, f_{-1})^{\ell} f_{jk} \mid 0 \leq \ell < j, 1 \leq k \leq c(j)\}$$
$$= \left\{f_{jk},\ [f_{-1}, f_{jk}],\ [f_{-1}, [f_{-1}, f_{jk}]],\ [f_{-1}, [f_{-1}, [f_{-1}, f_{jk}]]], \ldots, (\mathrm{ad}\, f_{-1})^{j-1} f_{jk})\right\}_{j\in\mathbb{N},\ k\leq c(j)}$$

This decomposition plays an important role in the task of associating a Lie group analog to $\mathfrak{m}$.

COMPLETION $\widehat{\mathfrak{m}}$ OF $\mathfrak{m}$ AND PRO-NILPOTENCE

The algebra $\mathfrak{m}$ also has the usual 'triangular decomposition'

$$\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

which is an eigenspace decomposition for the adjoint action, $\mathfrak{h}$ is the Cartan subalgebra and

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{m}_\alpha$$

We define the *(positive) formal completion* of $\mathfrak{m}$ to be

$$\widehat{\mathfrak{m}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}^+,$$

where

$$\widehat{\mathfrak{n}}^+ := \prod_{\alpha \in \Delta^+} \mathfrak{m}_\alpha = \prod_{k \in \mathbb{N}} \mathfrak{m}_k$$

and each $\mathfrak{m}_\alpha$ (or $\mathfrak{m}_k$) is *finite dimensional*.

Then $\widehat{\mathfrak{n}}^+$ is naturally pro-nilpotent, that is, an inverse limit of finite dimensional nilpotent Lie algebras.

Pro-summability

An infinite sum $\sum_i E_i$ of operators is called *summable* if

$$\sum_i E_i(y)$$

reduces to a finite sum for all y.

Let e_i be an imaginary simple root vector of $\mathfrak{m}$. Then $\exp(\mathrm{ad}(e_i))$ is an infinite sum.

However, we say that $\exp(\mathrm{ad}(e_i))$ is *pro-summable* since for all $y \in \widehat{\mathfrak{n}}^+$ $(\exp(\mathrm{ad}(e_i))(y)$ reduces to a finite sum (of unbounded length) when restricted to any component $\mathfrak{m}_k$ of

$$\widehat{\mathfrak{n}}^+ = \prod_{k \in \mathbb{N}} \mathfrak{m}_k.$$

This allows us to construct a group of pro-summable automorphisms of $\widehat{\mathfrak{n}}^+$.

Summary of main results - Adjoint group

With E. Jurisich and S. H. Murray

We have constructed a 'complete pro-unipotent group'

$$\widehat{\mathcal{U}} \quad \leq \quad \mathrm{Aut}\,(\widehat{\mathfrak{m}})$$

of automorphisms of the completion $\widehat{\mathfrak{m}}$ of $\mathfrak{m}$. The generators of $\widehat{\mathcal{U}}$ as a topological group are power series with constant term 1.

Elements of $\widehat{\mathcal{U}}$ are pro-summable of the form $g = \prod_{k=1}^{\infty} \exp\,(\mathrm{ad}\,(x_k))$ for $x_k \in \widehat{\mathfrak{n}}_k = \prod_{i \geq k} \mathfrak{m}_i$.

Any element $x \in \widehat{\mathfrak{m}}$ is an infinite (formal) sum, but $g \cdot x$ is finite (of unbounded length) when restricted to any component $\mathfrak{m}_k$ of $\widehat{\mathfrak{m}}$.

For Kac–Moody algebras, complete pro-unipotent groups have been constructed by Kumar and Rousseau.

SUMMARY OF MAIN RESULTS - GENERATORS AND RELATIONS

With E. Jurisich and S. H. Murray

We have constructed a Lie group analog $G(\mathfrak{m})$ for $\mathfrak{m}$ given by generators and relations.

Our group contains generators associated with imaginary roots of $\mathfrak{m}$ and relations between them.

The group $G(\mathfrak{m})$ has subgroups U_+^{im} and U_-^{im} generated by additive abelian groups, isomorphic to $(\mathbb{C}, +)$, one for each imaginary root.

The subgroups $U_+^{im}(\mathbb{C})$ and $U_-^{im}(\mathbb{C})$ have countably infinitely generated free subgroups $U_+^{im}(\mathbb{Z})$ and $U_-^{im}(\mathbb{Z})$. (*c.f.* The free Lie algebras $\mathfrak{u}_+^{im}$ and $\mathfrak{u}_-^{im}$).

The subgroup U_+^{im} of $G(\mathfrak{m})$ generated by all positive imaginary root groups embeds in $\widehat{\mathcal{U}} \leq \mathrm{Aut}\,(\widehat{\mathfrak{m}})$.

Summary of main results - Explicit automorphisms of $\mathfrak{m}$ and $\mathbf{M}$

With H. Chen, E. Jurisich and S. H. Murray

We have constructed both automorphisms of $\mathfrak{m}$ explicitly, giving a representation of $\mathbf{M}$ in $\mathrm{Aut}(\mathfrak{m})$.

The smallest permutation representation of $\mathbf{M}$ is permutations on

$$d = 97239461142009186000 = 2^4 \cdot 3^7 \cdot 5^3 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 29 \cdot 41 \cdot 59 \cdot 71$$

points. Let

$$c(15) = 126142916465781843075 = 3^6 \cdot 5^2 \cdot 7 \cdot 1483 \cdot 666739430527$$

This is one of the coefficients of q for the modular function

$$j(q) - 744 = \sum_{i \geq -1} c(i) q^i = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

with $c(-1) = 1, c(0) = 0, c(1) = 196884, \ldots$

The coefficients $c(j)$ appear as the dimensions of imaginary root spaces of $\mathfrak{m}$.

Permutation representation of $\mathbf{M}$

We have an embedding of the Monster finite simple group $\mathbf{M}$ in $\mathrm{Aut}\,(\mathfrak{m})$ via the smallest permutation representation of $\mathbf{M}$:

$$\mathbf{M} \le S_d \le S_{c(15)} \le \mathrm{Aut}\,(\mathfrak{m})$$

This gives a representation of $\mathbf{M}$ in terms of permutations

$$\sigma_{(k,k')} \in \mathrm{Aut}\,(\mathfrak{m})$$

for $1 \le k, k' \le c(15)$, where $\sigma_{(k,k')} \in S_{c(15)}$ permutes the indices on simple imaginary root vectors e_{jk}.

We seek a more concrete representation of $\mathbf{M}$ in $\mathrm{Aut}\,(\mathfrak{m})$. For this we must use the vertex operator algebra construction of $\mathfrak{m}$.

Thank you!