The moduli spaces $\mathcal{M}_g$ and $\mathcal{A}_g$: classical and tropical

Melody Chan
Brown University

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University of Warwick

Main theorem 1 of the talk (MC, Søren Galatius, Sam Payne)

$$\dim H^{4g-6}(\mathcal{M}_g; \mathbb{Q}) >> 1.32^g.$$

$\mathcal{M}_g$ is the **moduli space of Riemann surfaces of genus** g**.**

Main theorem 2 of the talk, time permitting (Madeline Brandt, Juliette Bruce, MC, Margarida Melo, Gwyneth Moreland, Corey Wolfe)

$$\mathrm{Gr}_0^W H_c^k(\mathcal{A}_5;\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } k = 10, 15, \\ 0 \text{ else}, \end{cases}$$

$$\mathrm{Gr}_0^W H_c^k(\mathcal{A}_6;\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } k = 12, \\ 0 \text{ else}, \end{cases}$$

$$\mathrm{Gr}_0^W H_c^k(\mathcal{A}_7;\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } k = 14, 19, 23, 28 \\ 0 \text{ else}. \end{cases}$$

The theorem refers to the "weight 0, compactly supported $\mathbb{Q}$-cohomology" of the **moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g.**

q / p	0	1	2	3	4	5	6	7	8	9	10
21	0	0	0	0	0	0	0	$\mathbb{Q}$			
20	0	0	0	0	0	0	0	0			
19	0	0	0	0	0	0	0	0			
18	0	0	0	0	0	0	0	0			
17	0	0	0	0	0	0	0	0			
16	0	0	0	0	0	0	0	$\mathbb{Q}$			
15	0	0	0	0	0	0	0	0			
14	0	0	0	0	0	0	0	0			
13	0	0	0	0	0	0	0	0			
12	0	0	0	0	0	0	0	$\mathbb{Q}$			
11	0	0	0	0	0	0	0	0			
10	0	0	0	0	0	$\mathbb{Q}$	0	0			
9	0	0	0	0	0	0	0	0			
8	0	0	0	0	0	0	0	0			
7	0	0	0	0	0	0	0	$\mathbb{Q}$			
6	0	0	0	0	0	0	$\mathbb{Q}$	0			
5	0	0	0	0	0	$\mathbb{Q}$	0	0			
4	0	0	0	0	0	0	0	0	0		
3	0	0	0	$\mathbb{Q}$	0	0	0	0	0		
2	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

Table: The (p, q) entry shows $\mathrm{Gr}_0^W H_c^{p+q}(\mathcal{A}_p; \mathbb{Q})$. The blank entries for $p \geq 8$ are currently unknown.

Part I. Moduli spaces

A **moduli space** is a parameter space.

Its points correspond to the geometric objects you want to study.

A moduli space is like a mail-order catalog. Pointing to the catalog specifies an object, elsewhere in a showroom.

Warmup: what is a moduli space of lines in $\mathbb{R}^2$?

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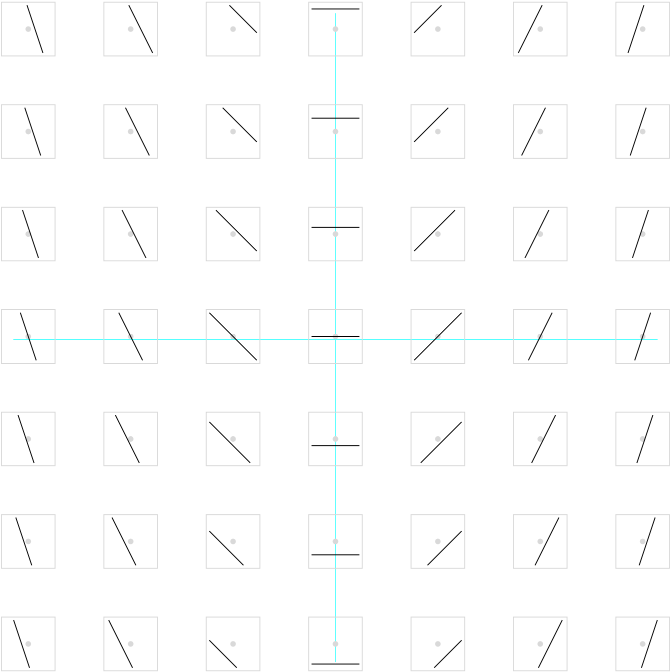
$$y = mx + c$$

$\mathbb{R}^2 = \{(m, c)\}$ is a moduli space for lines in $\mathbb{R}^2 \ldots$

$\ldots$ that are not vertical.



Next, $\mathbb{R}$ is a moduli space for vertical lines in $\mathbb{R}^2$:

$$\alpha \in \mathbb{R} \qquad \text{corresponds to} \qquad \text{the line } x = \alpha.$$

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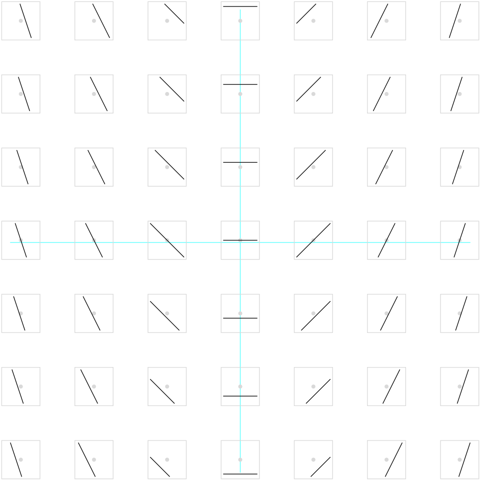
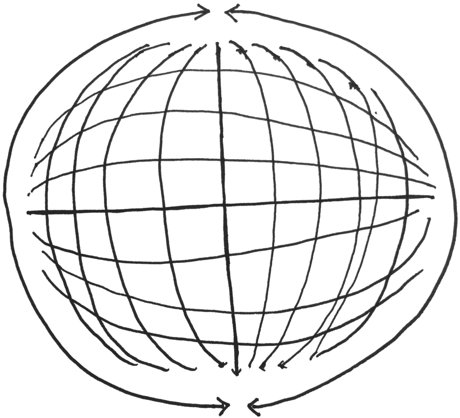
We wish to have a metric, or topology, on our moduli space, that expresses which objects are near each other.
We want to glue together

Notice:

The lines passing through a fixed point (x_0, y_0) form a line in the (m, c)-plane of slope $-x_0$.



This suggests a way to glue $\blacksquare \updownarrow$



We have constructed a moduli space of lines in $\mathbb{R}^2$!

Two distinct points in $\mathbb{R}^2$ determine a unique line.

now translates to

Two distinct lines in the (m, c)-plane meet at a unique point.

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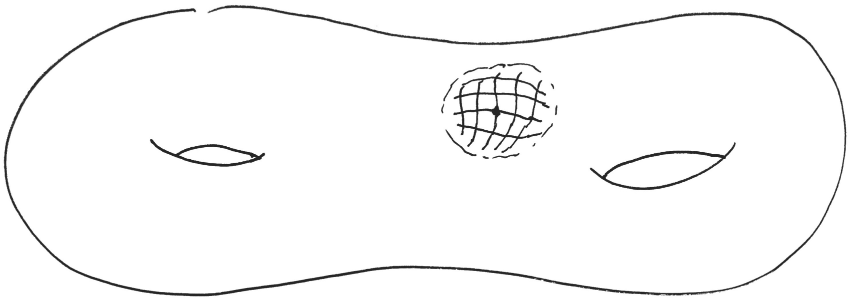
1. Intersections in the moduli space encode incidence problems.
2. This is one reason that *compactifications* of moduli spaces are helpful.
3. *Another reason to study compactifications: they can tell you about the topology of the space being compactified!*

Part II. $\mathcal{M}_g$

A **Riemann surface** is a compact, connected complex manifold of dimension 1.



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They are a meeting point for many different kinds of geometry (and algebra, combinatorics, physics...): we have identifications

1. isomorphism classes of Riemann surfaces of genus g
2. isomorphism classes of smooth, projective algebraic curves of genus g
3. isometry classes of hyperbolic surfaces of genus g

when $g \geq 2$.

The main character:

$$\mathcal{M}_g$$

the **moduli space of Riemann surfaces** of genus g, for $g \geq 2$.

$\mathcal{M}_g$ is a (variety/scheme/orbifold/Deligne-Mumford stack), irreducible of complex dimension $3g - 3$.

It was known in broad strokes already to Riemann, who coined the term **moduli**, in a paper in 1857.

But the formal construction followed much later, even after decades of studying $\mathcal{M}_g$ with the assumption that it could really be constructed!

(Grothendieck, Deligne-Mumford 60s)

Getting a feel for $\mathcal{M}_g$.

First recall: n**-dimensional projective space** is

$$\mathbb{P}^n = \{\text{lines in } \mathbb{C}^{n+1} \text{ through } 0\} = \{(z_0 : \cdots : z_n) : z_i \text{ not all } 0\}.$$

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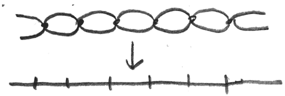
$\mathcal{M}_2$: every genus 2 curve admits a unique hyperelliptic involution...



and is determined by the arrangement of the 6 branch points on $\mathbb{P}^1$, up to isomorphism.

$$\mathcal{M}_2 = [\mathcal{M}_{0,6}/S_6] = \mathrm{UConf}(\mathbb{P}^1)/\operatorname{Aut}\mathbb{P}^1$$

$$\dim \mathcal{M}_2 = 6 - 3 = 3.$$

Getting a feel for $\mathcal{M}_g$.

$\mathcal{M}_3$: all (nonhyperelliptic) curves of genus 3 arise as smooth plane quartics.

A *smooth plane quartic curve* is the set of solutions in $\mathbb{P}^2$ to a homogeneous polynomial of degree 4 in x, y, z

$$a_{4,0,0}x^4 + a_{3,1,0}x^3y + \cdots + a_{0,0,4}z^4 = 0.$$

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The moduli space of smooth plane quartic curves is

$$\mathbb{P}^{14} - \Delta$$

where Δ is the discriminantal hypersurface, parametrizing those $(a_{4,0,0} : \cdots : a_{0,0,4}) \in \mathbb{P}^{14}$ that define *singular* plane curves.

$$\mathcal{M}_3 \dashleftarrow (\mathbb{P}^{14} - \Delta)/\mathrm{Aut}(\mathbb{P}^2)$$

$$\dim \mathcal{M}_3 = 14 - 8 = 6.$$

$\mathcal{M}_2$. Image: Alicia Harper

In this talk, I'll discuss **the rational cohomology of $\mathcal{M}_g$.** For $i \geq 0$,

$$H^i(\mathcal{M}_g; \mathbb{Q})$$

is a finite-dimensional vector space over $\mathbb{Q}$, measuring "the space of holes in dimension i."

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Equivalently, $\mathcal{M}_g = \mathcal{T}_g / \mathrm{Mod_g}$ is the quotient of Teichmüller space by the *mapping class group* $\mathrm{Mod_g}$. Therefore we equivalently study the **cohomology of the mapping class group.**

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Roughly speaking, the cohomology of $\mathcal{M}_g$, and its compactifications, is studied in analogy to arithmetic groups (Borel etc.), and to Grassmannians (Littlewood-Richardson etc.)

How much cohomology is there?

Harer-Zagier 1986: Asymptotically,

$$\chi(\mathcal{M}_g) = \dim H^0(\mathcal{M}_g;\mathbb{Q}) - \dim H^1(\mathcal{M}_g;\mathbb{Q}) + \dots$$

grows superexponentially in g:

$$(-1)^{g+1}\chi(\mathcal{M}_g) \sim g^{2g}.$$

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grows superexponentially in g:

$$(-1)^{g+1}\chi(\mathcal{M}_g) \sim g^{2g}.$$

But we know only a vanishingly small proportion of the cohomology *explicitly*.

In what range does cohomology appear?

- In degrees at most $4g-6$ (Harer, Church-Farb-Putman, and Morita-Sakasai-Suzuki).

- Moreover, conjectures in the literature had implied that $H^{4g-6-i}(\mathcal{M}_g;\mathbb{Q}) = 0$ for any fixed $i \geq 0$, for $g >> 0$. (Church-Farb-Putman 2012, and Kontsevich 1993)

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Our theorem

$$\dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) >> 1.32^g$$

finds cohomology in highest possible degree, and refutes both those conjectures.

Ingredients for proof that

$$H^{4g-6}(\mathcal{M}_g; \mathbb{Q}) >> 1.32^g.$$

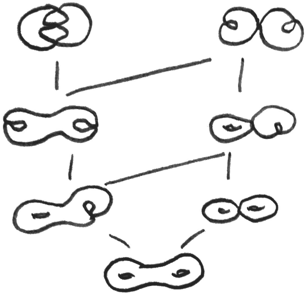
1. The Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$.
2. Tropical geometry/tropical moduli spaces of curves.
3. Kontsevich's graph complex and theorems of Willwacher from quantum algebra.

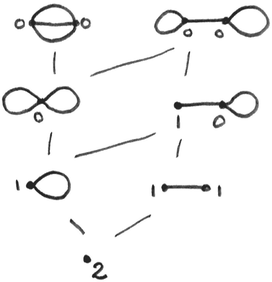
1. The Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$.

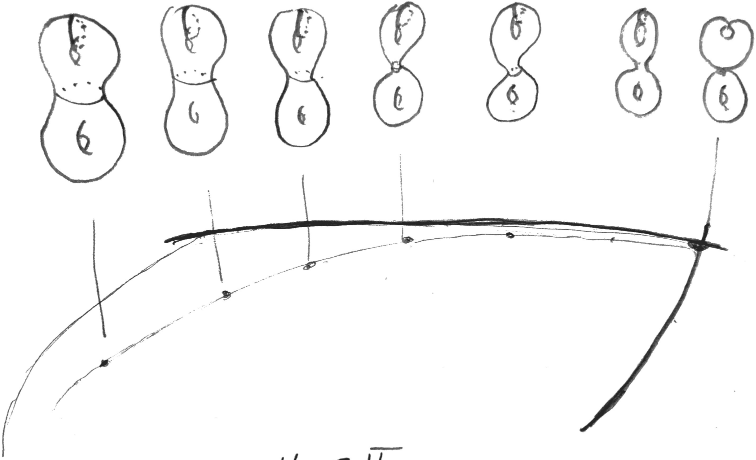
$\mathcal{M}_g$ is not compact. In an influential 1969 paper, Deligne-Mumford constructed a compactification $\mathcal{M}_g \subset \overline{\mathcal{M}_g}$, the moduli space of *stable curves* of genus g.

Definition. A genus g **stable curve** is a smooth or nodal complex algebraic curve, of arithmetic genus g, having only finitely many automorphisms.

Stable curves come in finitely many topological types, equivalently dual graphs.







$\mathcal{M}_2 \subset \overline{\mathcal{M}_2}$

2. Tropical geometry/tropical moduli spaces of curves.

Tropical geometry is a modern degeneration technique in algebraic geometry—one in which the limiting object is entirely combinatorial.

To get the flavor, consider the family of projective plane quartics C_t, parametrized by $t \in \mathbb{C}$, defined by the equation

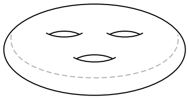
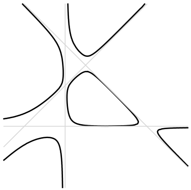
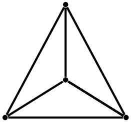
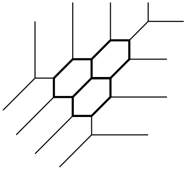
$$t(x^4 + y^4 + z^4) + xyz(x + y + z) = 0. \qquad (1)$$

When $t \to 0$, the curve degenerates to the zero locus of

$$xyz(x + y + z) = 0.$$

Figure: Left: $\mathcal{C}_0$. Right: $\mathrm{Trop}(\mathcal{C}_0)$.

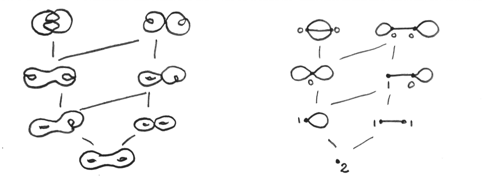
Table: Cartoons of abstract/embedded algebraic/tropical curves of genus 3.

	abstract	embedded
algebraic		
tropical		

The main input from tropical geometry for today is the **tropical moduli space of curves** Δ_g.

(Brannetti-Melo-Viviani, Caporaso, Gathmann-Markwig, Culler-Vogtmann,...)

Every stable curve in $\overline{\mathcal{M}_g}$ has a vertex-weighted dual graph.



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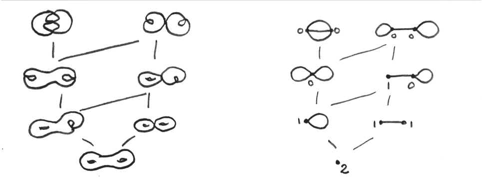


Definition. A **tropical curve** of genus g is a vertex-weighted dual graph G arising in this way, together with *any* metrization $\ell \colon E(G) \to \mathbb{R}_{>0}$ with total length 1.

Definition. Let $\boldsymbol{\Delta}_{\mathbf{g}}$ denote the moduli space of genus g normalized tropical curves.

Remark: The tropical moduli space Δ_g arises in several different geometric contexts.

- the quotient of Harvey's curve complex on S_g by Mod_g
- the simplicial completion of $X_g/\mathrm{Out}\,F_g$, where X_g denotes Culler-Vogtmann Outer Space
- up to homotopy (CGP), the one point compactification $(X_g/\mathrm{Out}\,F_g)^*$

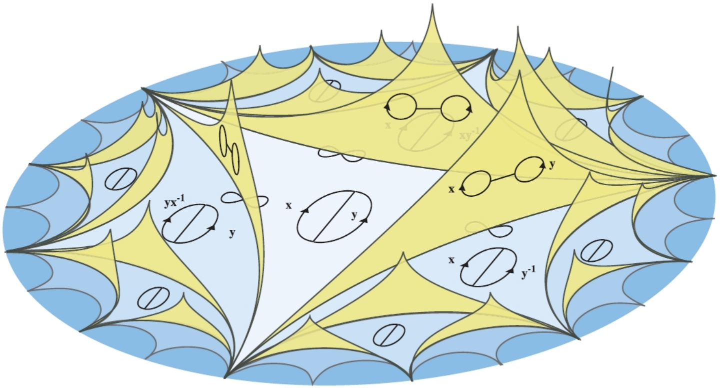
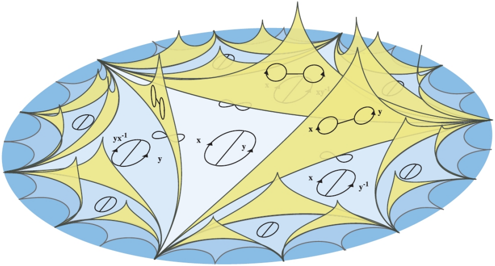


Figure 2: Outer space in rank 2

(Vogtmann “What is Outer Space?” AMS Notices August 2008)



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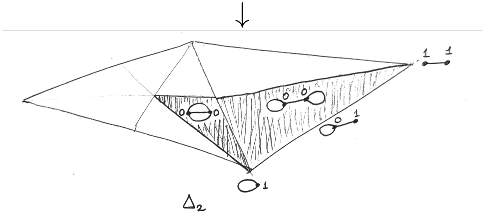

Deligne's theory of mixed Hodge structures implies:

$$H^{2d-i}(\mathcal{M}_g; \mathbb{Q}) \twoheadrightarrow H_{i-1}(\Delta_g; \mathbb{Q}).$$

The cohomology groups of $\mathcal{M}_g$ *surject* onto the homology groups of Δ_g, with degree shift.

The main technical result of CGP gives an isomorphism between the homology of Δ_g to the homology of **Kontsevich's 1994 graph complex**

$$\cdots \to G_i^g \to G_{i-1}^g \to G_{i-2}^g \to \cdots.$$

Here G_i^g are finite dimensional vector spaces spanned by certain **graphs** of genus g with i edges.

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Willwacher (2015) and F. Brown (2012) prove remarkable theorems about the graph complex, coming from quantum algebra/number theory, from which our theorem is deduced.

Even though computer calculations don't appear in our paper, they were crucial to finding the right theorem.

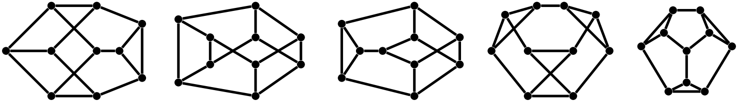


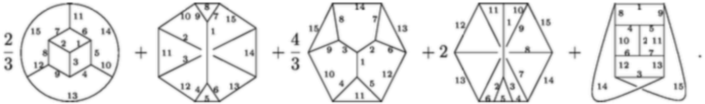
Figure: The graphs appearing in the unique nonzero reduced homology class in Δ_6, with unsigned coefficients $2, 3, 6, 3, 4$.

Habitat. While simplest to define, Basic Graph Cohomology does not appear in nature.

Results. At present, very little is known about ${}^{bc}H^k_n$. The only dimensions we have computed are in Table 1. The data in that table is displayed using the folowing format for each pair (n, k):

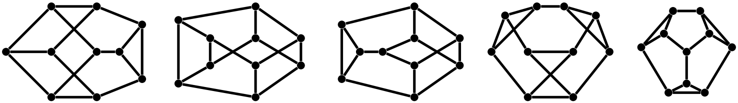
(2)
$$\boxed{\begin{array}{lr} \mathbf{dim}\ {}^{bc}\boldsymbol{H^k_n} & \dim {}^{bc}C^k_n \\ \multicolumn{2}{l}{\dim\ker d|_{{}^{bc}C^k_n} / \dim \operatorname{im} d|_{{}^{bc}C^{k-1}_n}} \end{array}}$$

Example 3.13. ${}^{bc}H^0_5$ is generated (over $\mathbb{Q}$) by



Problems. ${}^{b}H$ is simpler than its twist H, defined below. Why is it that H is related to so many things while ${}^{b}H$ is related to none? What is ${}^{b}H$?

(Bar-Natan–McKay 2001 "Graph cohomology")



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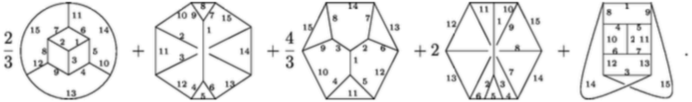

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	$n=4$	$n=5$	$n=6$	$k=7$	$n=8$	$n=9$
$k=0$	0	**1** 7 1/0	**0** 29 0 / 0	**0** 214 0 / 0	**0** 2496 0 / 0	**1** 30307 1 / 0
$k=1$	**0** 1 0/0	**0** 13 6 / 6	**0** 109 29/29	**0** 1261 214/214	**?** 16134 ? / 2496	**?** 226296 ? / 30306
$k=2$	**1** 2 2/1	**0** 12 7 / 7	**0** 186 80/80	**1** 2926 1048/1047	?	?
$k=3$		**0** 6 5/5	**0** 170 106/106	**0** 3491 1878/1878	?	?
$k=4$		**0** 1 1/1	**1** 75 65/64	**0** 2328 1613/1613	?	?
$k=5$			**0** 10 10/10	**0** 879 716/715	**?** 38906 27533/?	?
$k=6$				**0** 179 163/163	**1** 13867 11374/11373	?
$k=7$				**0** 16 16/16	**0** 2742 2493/2493	?
$k=8$					**0** 262 249/249	?
$k=9$					**0** 14 13/13	?
$k=10$					**0** 1 1/1	?

(Bar-Natan–McKay 2001 “Graph cohomology”)

What about the results I showed you for $H^*(\mathcal{A}_g; \mathbb{Q})$? This is the moduli space of prinicpally polarized abelian varieties of dimension g.

What about the results I showed you for $H^*(\mathcal{A}_g;\mathbb{Q})$? This is the moduli space of prinicpally polarized abelian varieties of dimension g.

Equivalently, since $\mathcal{A}_g = \mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z})$,

$$H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}).$$

Ingredients for proof:

1. Compactification
2. Tropicalization
3. Extraction of a combinatorial chain complex

Ingredients for proof:

1. Toroidal compactifications of $\mathcal{A}_g$ (Ash-Mumford-Rapaport-Tai), specifically the *perfect cone compactification*.
2. Tropical moduli spaces of abelian varieties A_g^{trop}. (Brannetti-Melo-Viviani, Mikhalkin-Zharkov)
3. The perfect cone complex $P_\bullet^{(g)}$ (BBCMMW).

A brief remark on $P_\bullet^{(g)}$. The moduli space A_g^{trop} has a stratification

$$A_g^{\mathrm{trop}} = Q_0 \sqcup Q_1 \sqcup \cdots \sqcup Q_g$$

where $Q_h = \{\text{positive definite } h \times h \text{ matrices}\}/\mathrm{GL}_h(\mathbb{Z})$.

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One of the main technical theorems of BBCMMW constructs a short exact sequence

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$V_\bullet^{(g)}$ was studied/computed by Elbaz-Vincent-Gangl-Soulé. Our computations use the computations of EVGS as input.

Thank you! (Image: A. Harper)