Finite Element Approximation of Ricci Curvature and Simulation of Ricci-DeTurck Flow

Hans Fritz
(University of Freiburg)

The Ricci Flow is a partial differential equation deforming a time-dependent metric $g(t)$ on a closed, n-dimensional manifold $\mathcal{M}$ proportional to its Ricci curvature. It was introduced by Richard Hamilton in 1982 as an approach towards the geometrisation conjecture. In 2002, Hamilton's programme was finally completed by Grigori Perelman in a series of breathtaking papers.
In contrast to its impact on pure mathematics, the Ricci Flow has not attracted appropriate attention in Numerics. The hitherto existing numerical studies of the Ricci Flow are either restricted to two-dimensional manifolds or to manifolds of revolution. In this talk we present a numerical method for the simulation of the n-dimensional Ricci-DeTurck Flow. This flow is a reparametrisation of the Ricci Flow by a smooth family of diffeomorphisms that solves the Harmonic Map Flow. Our algorithm is based on the assumption that the manifold $\mathcal{M}$ admits a codimension one embedding $F : \mathcal{M} \to \mathbb{R}^{n+1}$ into an Euclidean space, where the embedding is not supposed to be isometric with respect to the Euclidean metric. We then introduce a Riemannian metric $G(t)$ on a tubular neighbourhood $\Gamma_\delta \subset \mathbb{R}^{n+1}$ of the embedding $\Gamma := F(\mathcal{M})$ such that $(\Gamma, g(t))$ is a Riemannian submanifold of $(\Gamma_\delta, G(t))$. By this means, the Ricci-DeTurck Flow can be formulated as an evolution equation for the metric $G(t)$. In particular, it is possible to replace the local equations for $g_{ij}(t)$ by global equations for the Cartesian components $G_{\alpha\beta}(t)$ of the metric $G(t)$. The discretisation of the weak formulation of these global equations with Surface Finite Elements leads to an algorithm for the computation of the metric $G(t)$ on $\Gamma \times [0, T)$. All geometric quantities in this algorithm can be computed using the Cartesian coordinates of the ambient space. Using piecewise linear Finite Elements, experimental convergence tests for two- and three-dimensional hypersurfaces show the convergence of our algorithm. Further, the algorithm seems to work well also for the simulation of finite-time singularities like neck pinch singularities.
So far, we have proved consistency estimates for our definition of the discrete Ricci curvature in the case that the embedding of the Riemannian manifold is isometric with respect to the Euclidean metric. This definition is based on a Finite Element discretisation of Yano's weak formulation of the Ricci curvature. This means that we define the discrete Ricci curvature of a polyhedral hypersurface $\Gamma_h \subset \mathbb{R}^{n+1}$ of arbitrary dimension n as the $L^2(\Gamma_h)$-projection of a discretised curvature functional. For a piecewise quadratic approximation Γ_h of a two- or three-dimensional hypersurface $\Gamma \subset \mathbb{R}^{n+1}$ this definition approximates the Ricci curvature of Γ with a linear order of convergence in the $L^2(\Gamma)$-norm. By using a discrete smoothing scheme in the case of a piecewise linear approximation of Γ, we still get a convergence of order $\frac{2}{3}$ in the $L^2(\Gamma)$-norm and of order $\frac{1}{3}$ in the $W^{1,2}(\Gamma)$-norm.