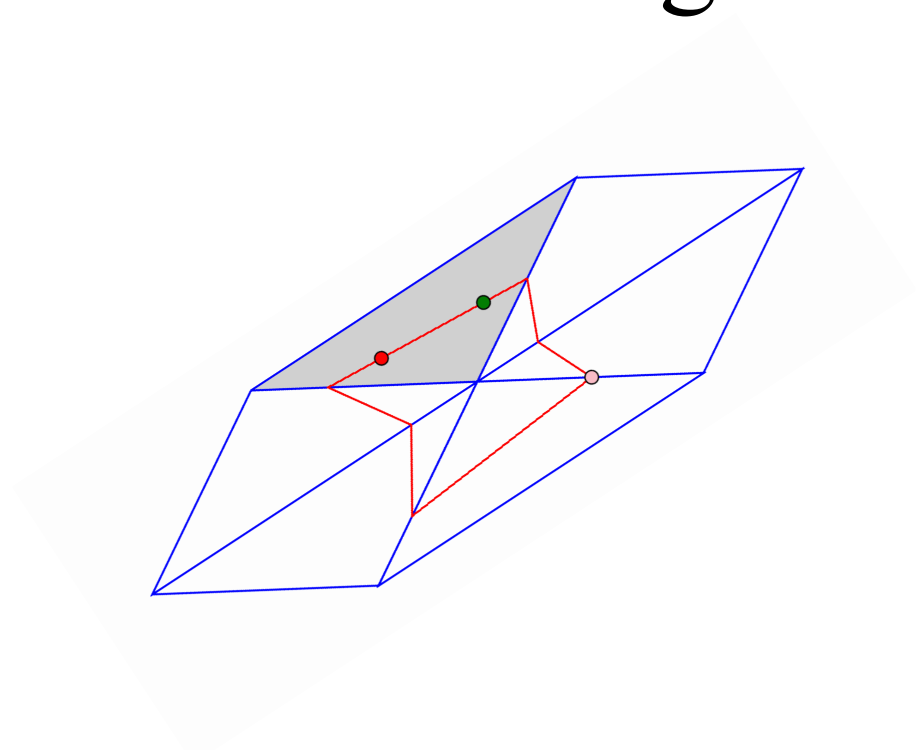
Introduction to tiling billiards



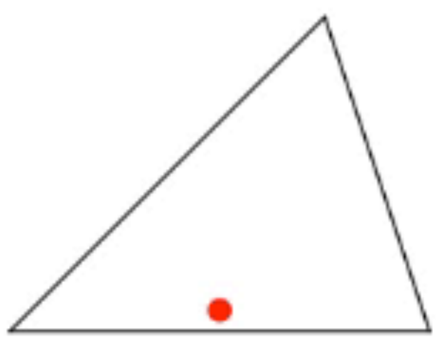
Olga Paris-Romaskevich
CNRS, Institut de Mathématiques de Marseille

for *Ergodic theory and Dynamical Systems at Warwick*

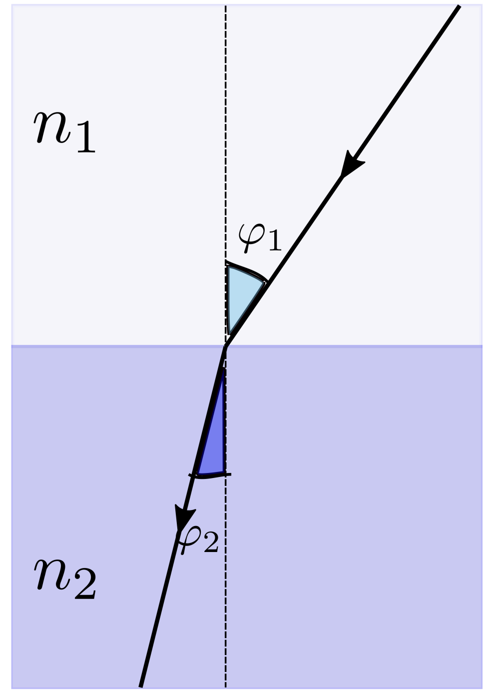
8 December 2020

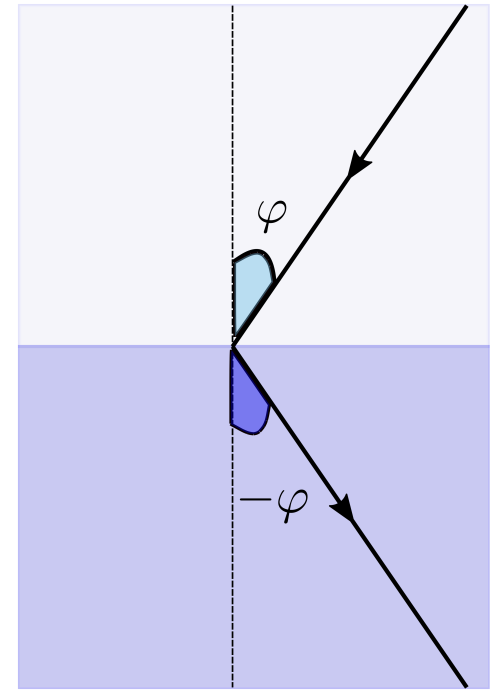
TILING BILLIARDS

Goal : understand the dynamics depending on the angles of a triangle and initial conditions

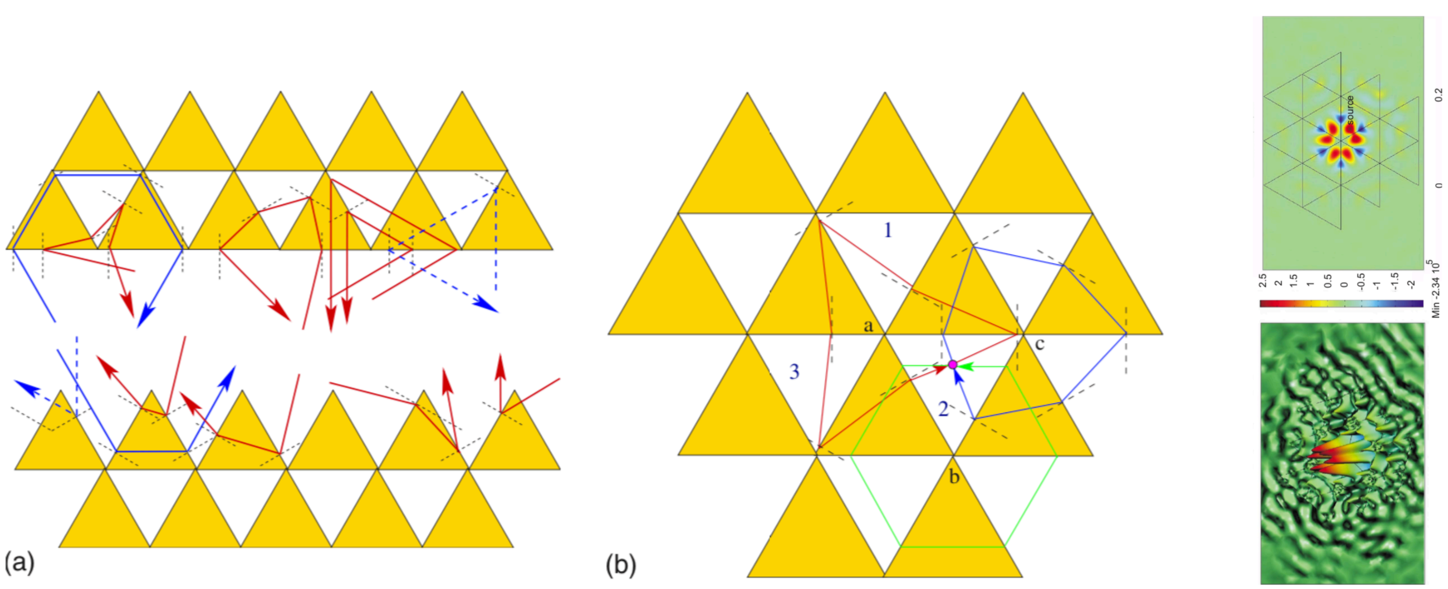


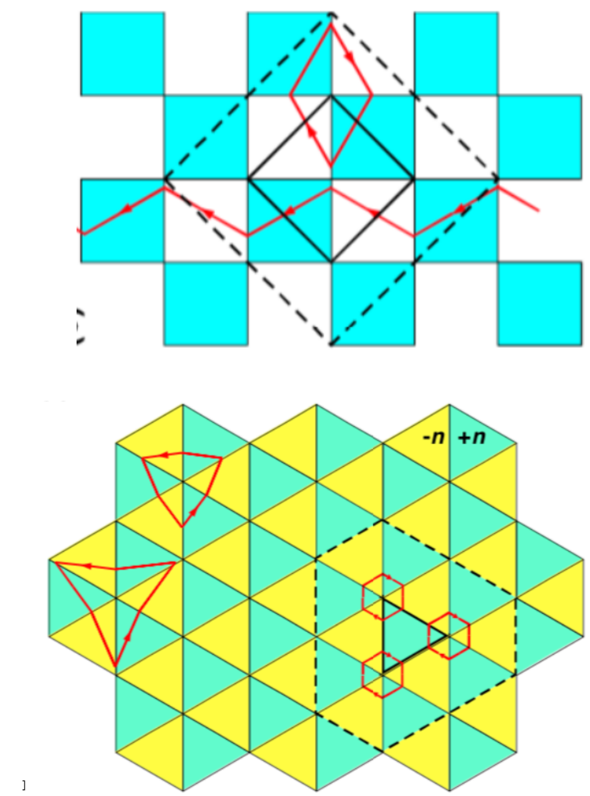
Naive physical motivation : movement of light in heterogeneous media





Snell's Law $\quad \dfrac{n_1}{n_2} = \dfrac{\sin\theta_2}{\sin\theta_1}$





© S. Anantha Ramakrishna, S. Guenneau, S. Enoch, G. Tayeb, B. Gralak *Confining light with negative refraction in checkerboard metamaterials and photonic crystals (2007)*

© Mascarenhas, B. Fluegel *Antisymmetry and the breakdown of Bloch's theorem for light* (2015)

Goal of the talk

I would like to introduce you to the simple and not-so-simple behaviour of these billiards. And also, present the ideas of different flavours in the research on tiling billiards, coming from origami, symbolic dynamics, topology, foliations theory and dynamics.

Tiling billiards is a relatively new subject, interest in it emerged only in 2016.

History and references

1. Diana Davis starts to study the system with a group of undergraduate students

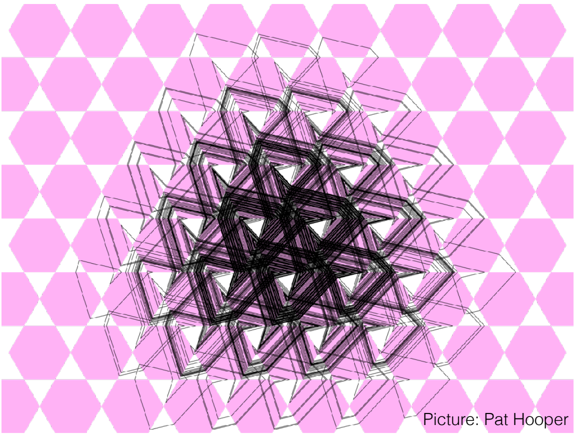
- D. Davis, K. DiPietro, J. Rustad, A. St Laurent *Negative refraction and tiling billiards ('16)*

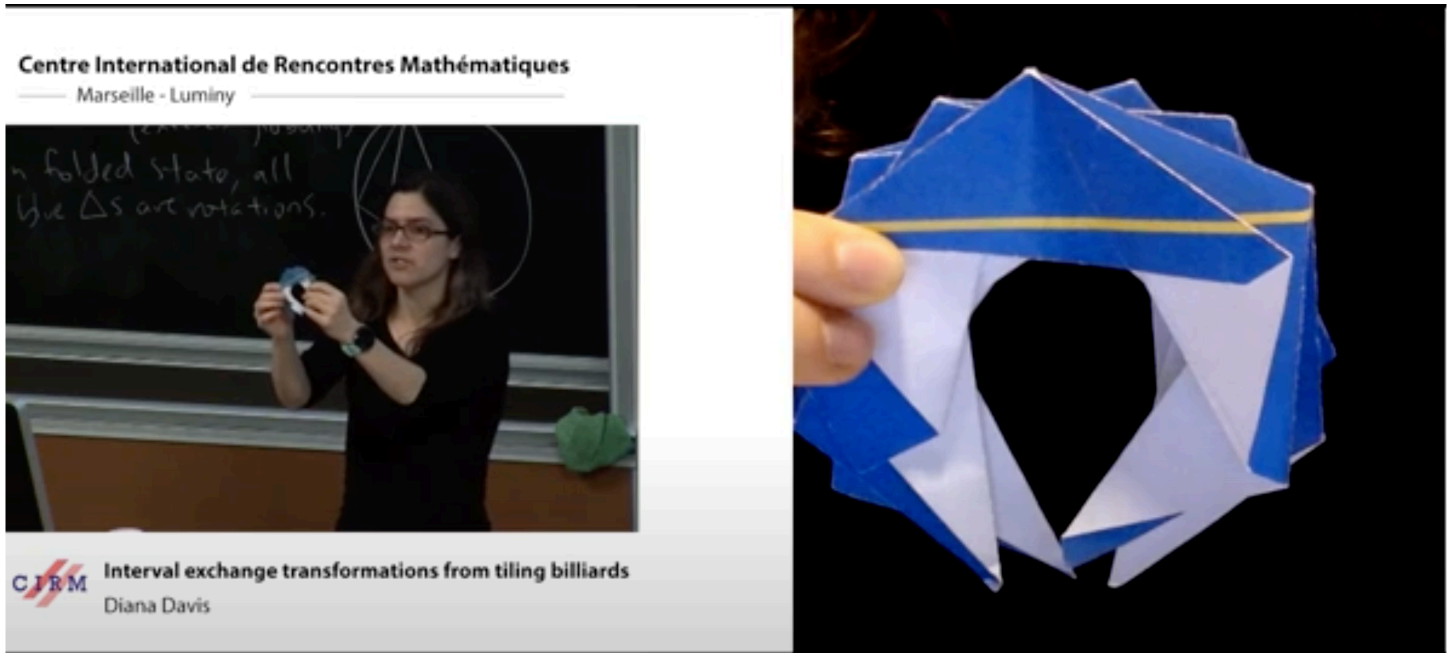
- P. Baird-Smith, D. Davis, E. Fromm, S. Iyer *Tiling billiards on triangle tilings, and interval exchange transformations ('17-'19)*

Connection with interval exchange transformations with flips is discovered !

2. Diana Davis gives a talk at a conference in Marseille, on 14th Feb. 2017.
This got me interested in tiling billiards!

- D. Davis, P. Hopper *Periodicity and ergodicity in the trihexagonal tiling ('16)*



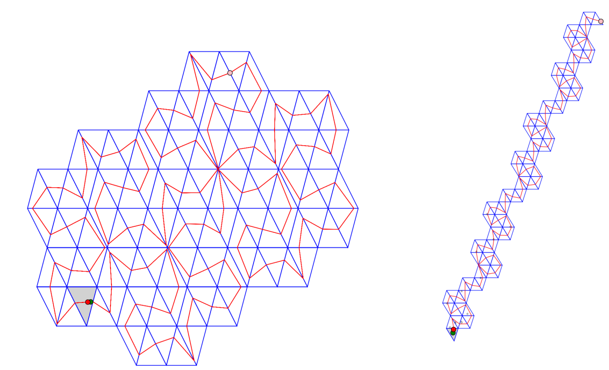


History and references

3. Description of generic behaviour for triangle tiling billiards.

- P. Hubert, O. Paris-Romaskevich *Triangle tiling billiards draw fractals only if aimed at the circumcenter* ('18-'19)

Generic behaviour is described and exceptional trajectories pointed out.
Connection with classical work by Arnoux and Rauzy is made precise !



4. Complete classification of dynamics and introduction of new tools

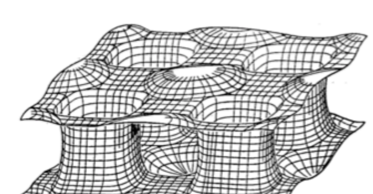
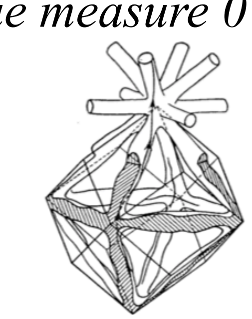
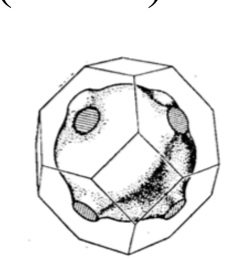
- O. Paris-Romaskevich *Trees and flowers on a billiard table* ('20), preprint

Classification of trajectories is proven.
Tiling billiard foliations and renormalisation are introduced for triangle tiling billiards.
The Tree Conjecture (P. Baird-Smith, D. Davis, E. Fromm, S. Iyer) is proven.
Connections with classical works by Hooper-Weiss, Rauzy and McMullen are pointed out.

5. True physical interpretation of tiling billiards : Novikov's problem on conductivity of metals ('82 !)

- Hubert&Mercat&Paris-Romaskevichand Dynnikov&Skripchenko *Novikov's gasket has zero Lebesgue measure 0* ('20-21)

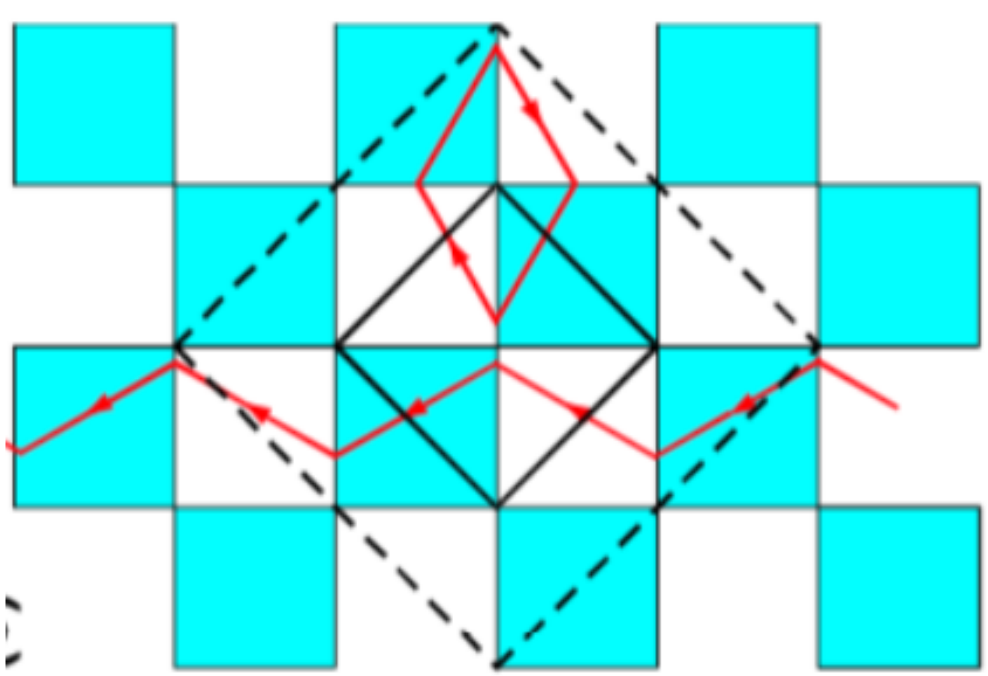
Tiling billiards (in triangle and cyclic quadrilateral tilings) shed new light on **Novikov's problem** and help to answer it in its loose form in genus 3.

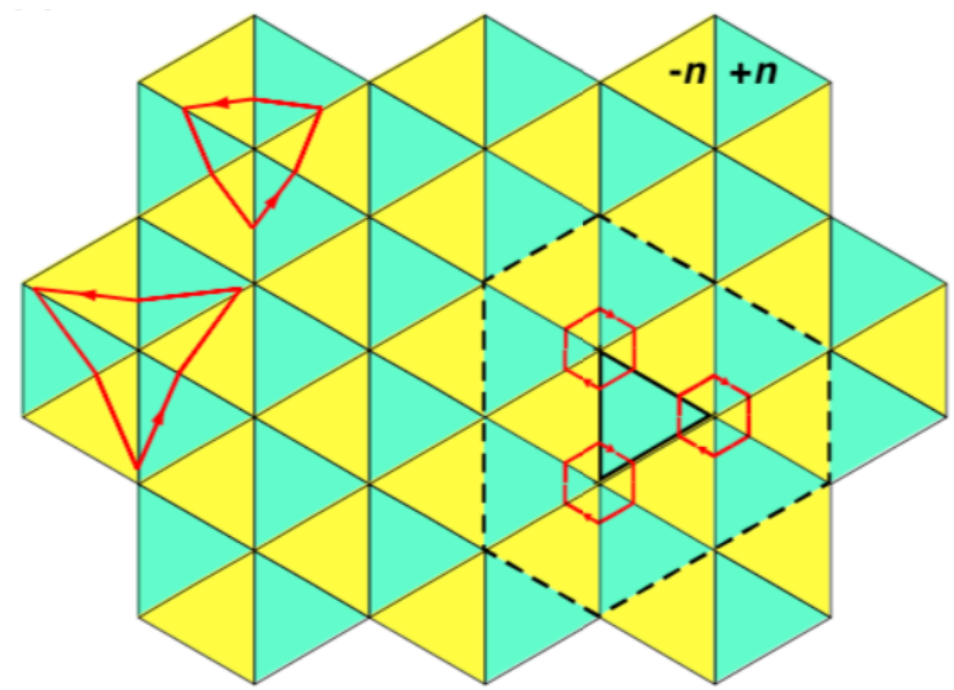




Plan of the talk

- examples of trajectories
- first idea : folding
- consequence of folding : connection with interval exchange transformations with flips and qualitative behaviour description
- second idea : tiling billiard foliations
- consequence of the existence of these foliations : singular trajectories and flowers
- example of an exceptional trajectory
- classification theorem for triangle tiling billiards
- Novikov's problem formulation

Examples of trajectories



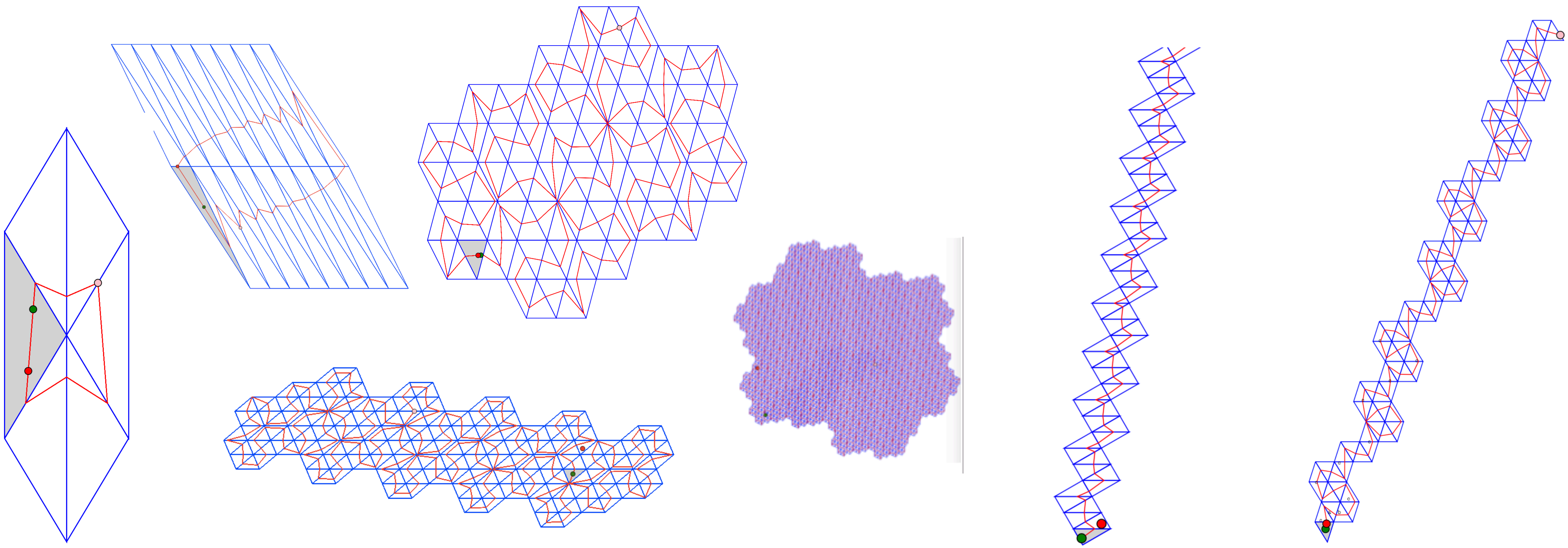


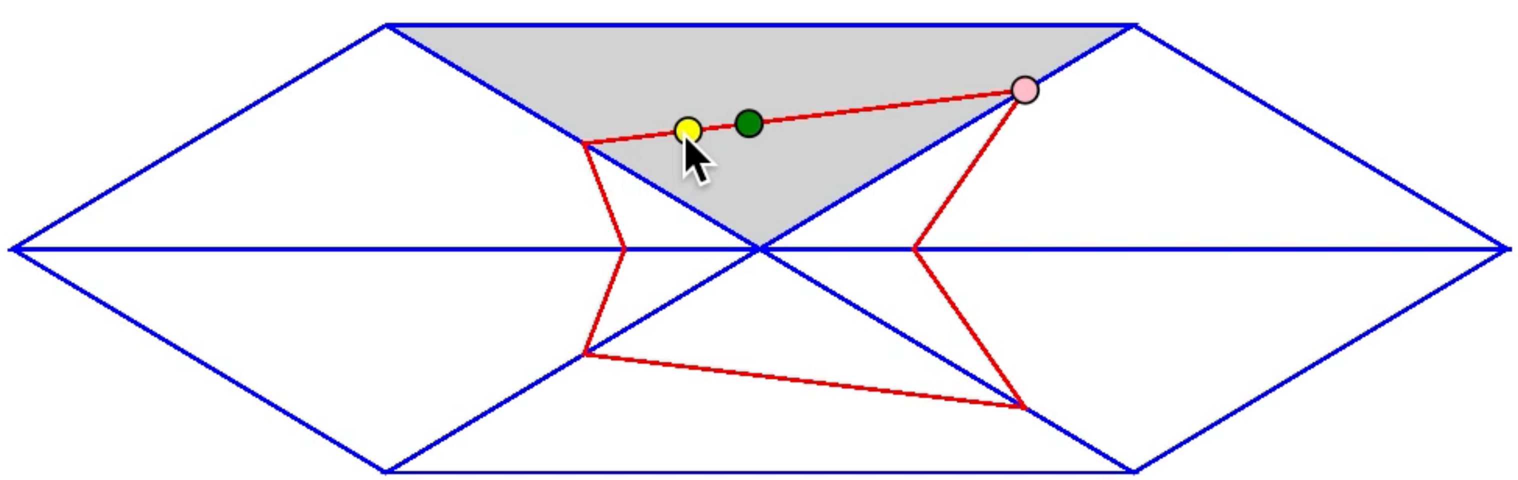
What happens if one replaces an equilateral triangle (square) by any triangle (quadrilateral)?

Are periodic trajectories still stable ?! Do they even exist ?

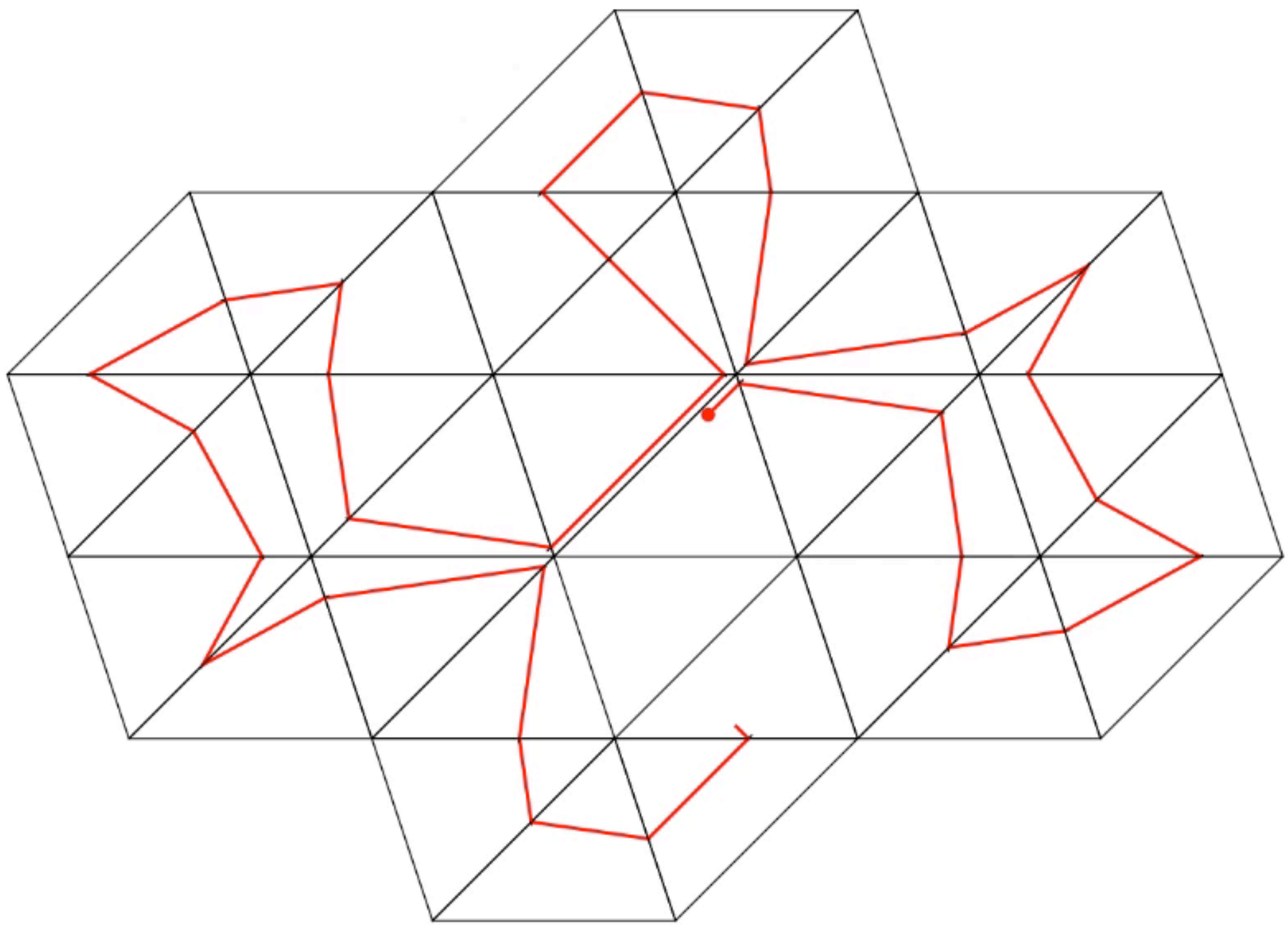
What do you think ?

Examples of trajectories

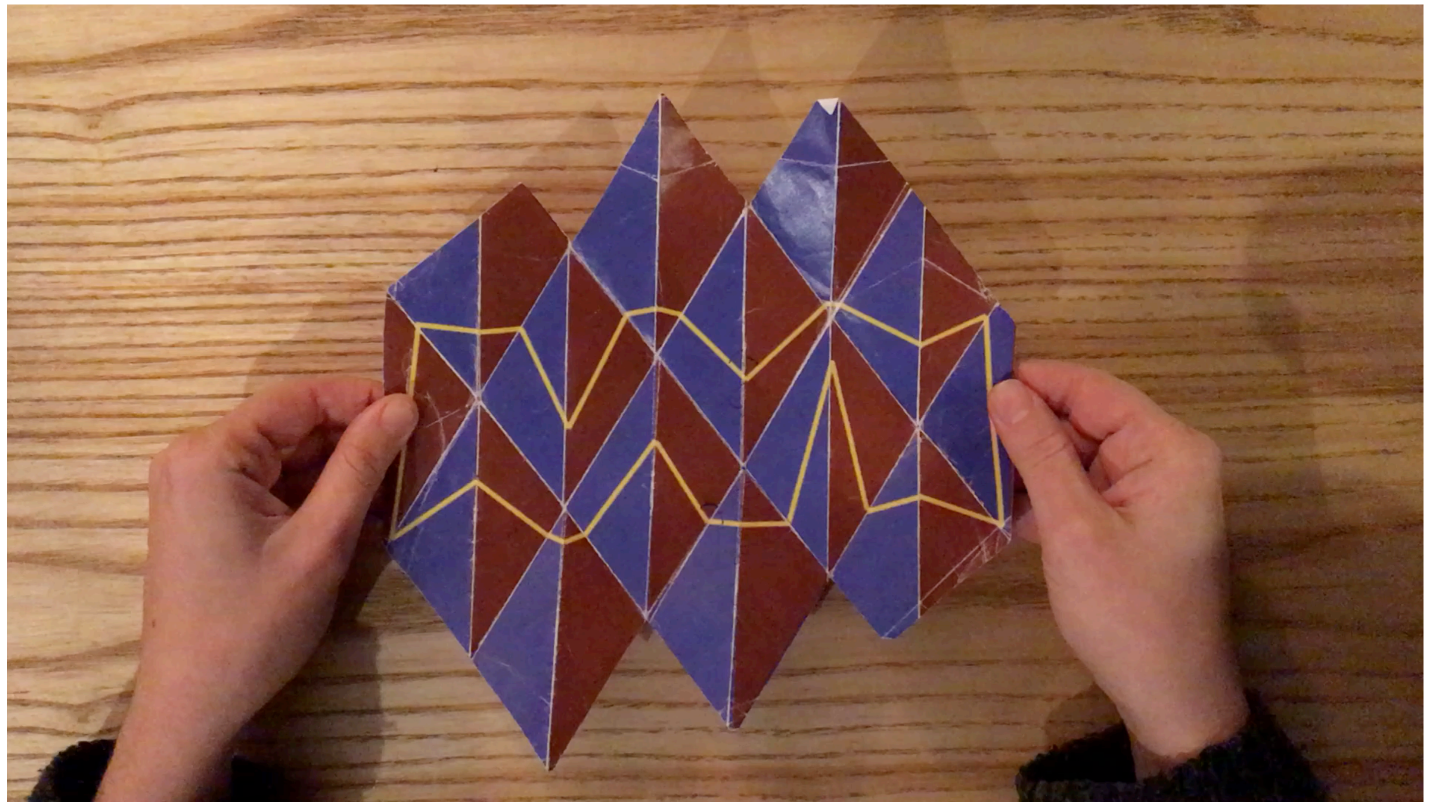




key property (and explanation to stability of periodic trajectories) — the trajectories fold into line segments or rays

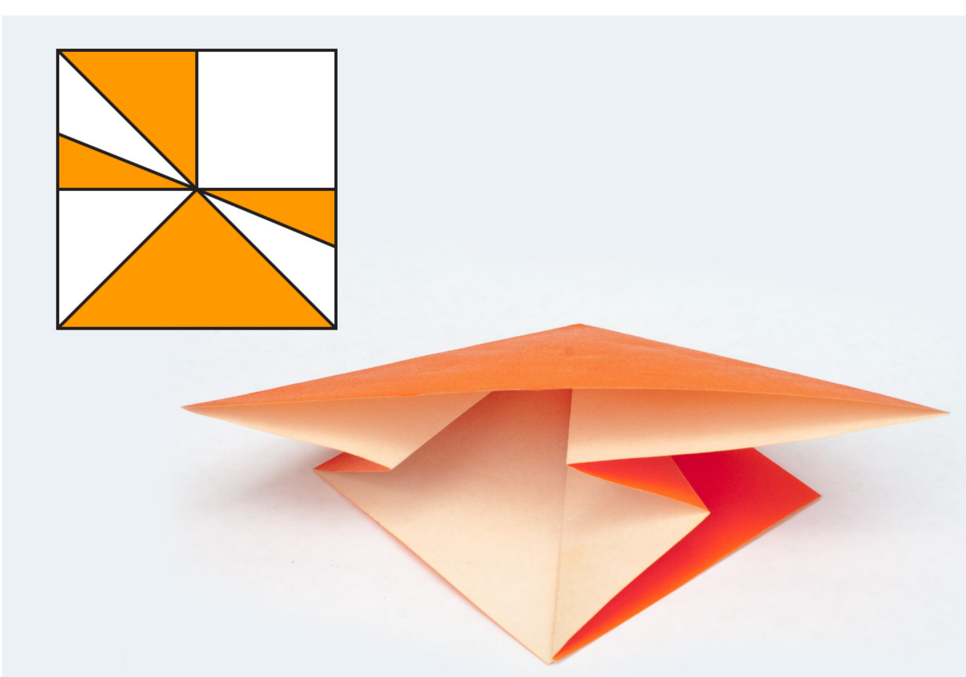


Theorem : the folding of such a triangle tiling is well defined globally !
(doesn't depend on the folding path)



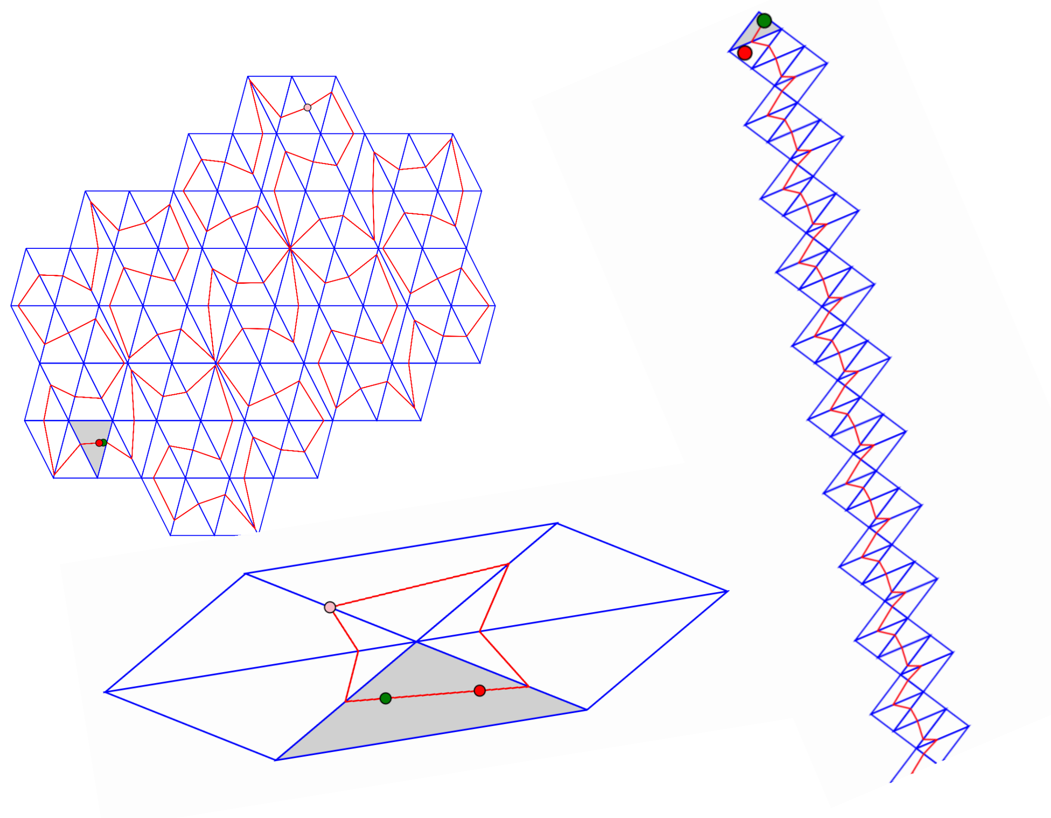
Remark for the specialists :the folding also shows why one has connection with interval exchange transformations with flips

Proof : along the lines of the Kawasaki's theorem



This property of *global foldability* is very important throughout all the talk, and gives simple and powerful consequences for the behaviour of the billiard.

Consequences of the key property for the dynamics.

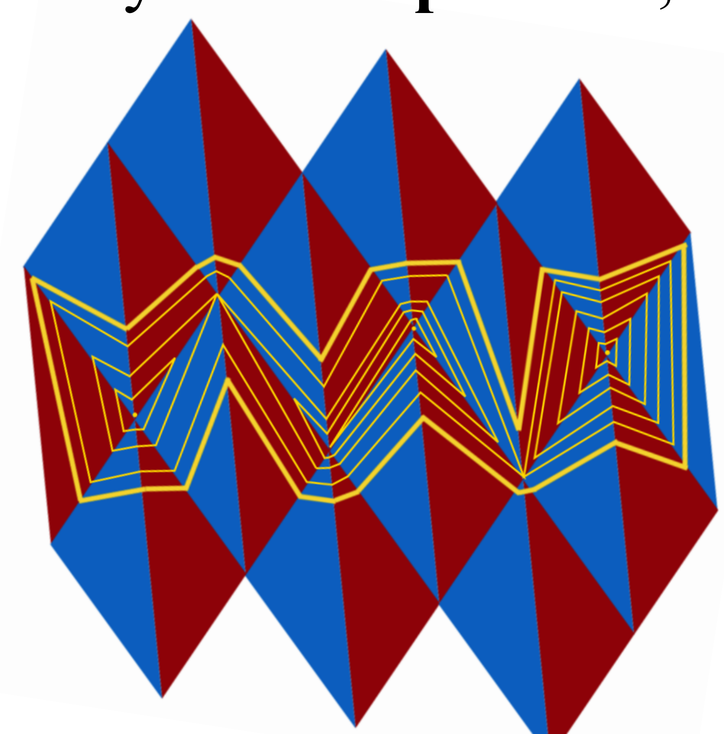


1. Any trajectory passes at most once by any tile
2. Any bounded trajectory is a simple closed curve
3. Distance to the circumcenter of a tile is an integral of motion

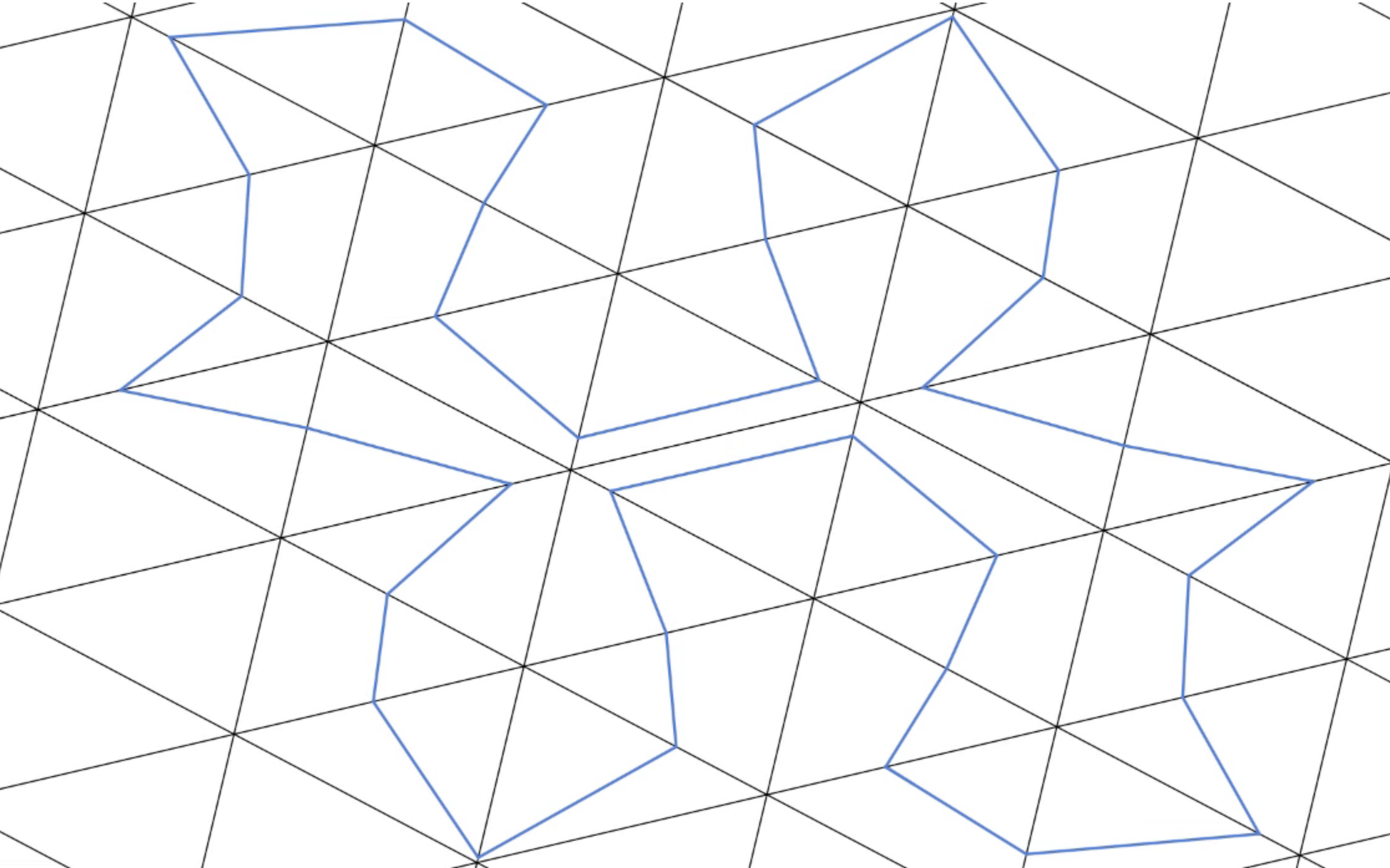
(P. Baird-Smith, D. Davis, E. Fromm, S. Iyer, 2017) Any trajectory is either **periodic**, or **linearly escaping**, or **non-linearly escaping**.

A much stronger result holds.
(P. Hubert, P.-R. 2019) For almost any triangle tiling, every trajectory is either **periodic**, or **linearly escaping**. Non-linearly escaping are exceptional.

4. (P.-R., 2020) Existence of parallel foliations by trajectories of a full plane



What does the existence of parallel foliations mean for a fixed periodic trajectory ? It means that it can be understood in terms of a finite number of singular trajectories (flowers).

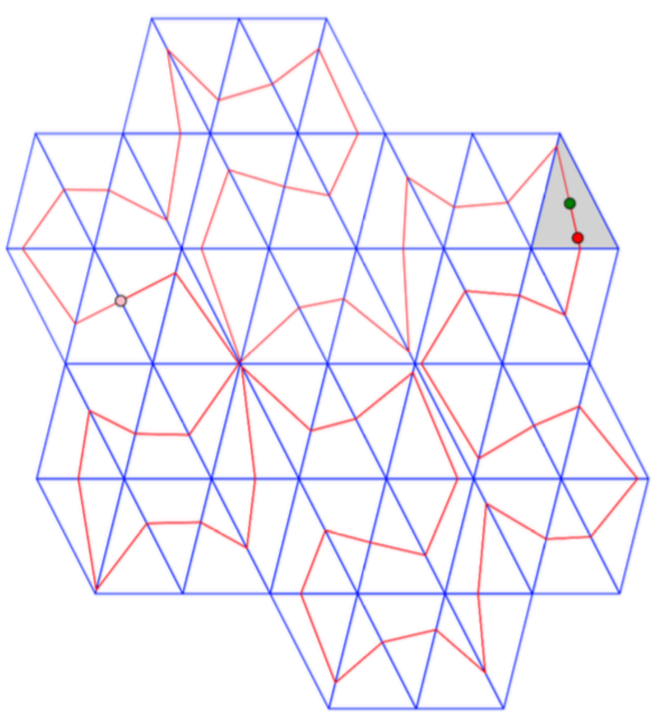


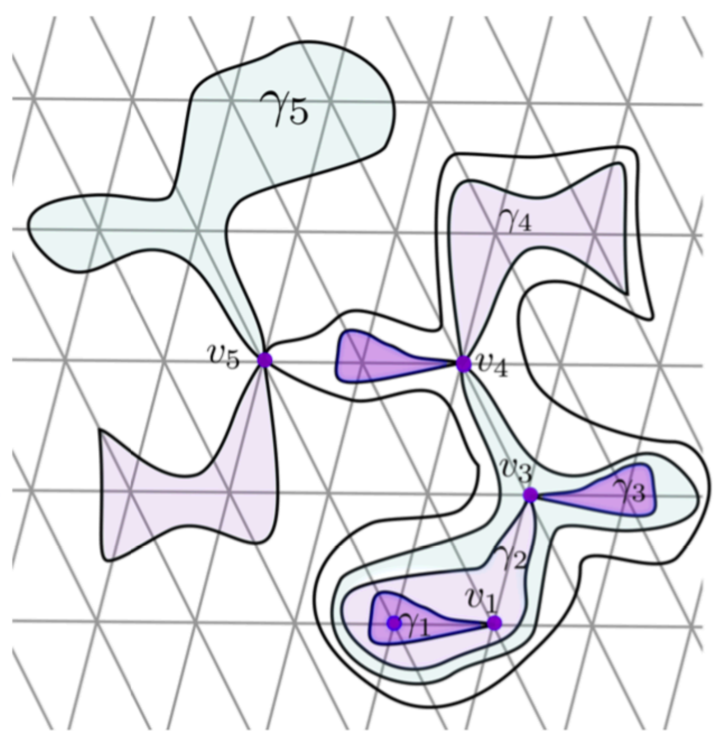
What about exceptional behaviour ? — one example.

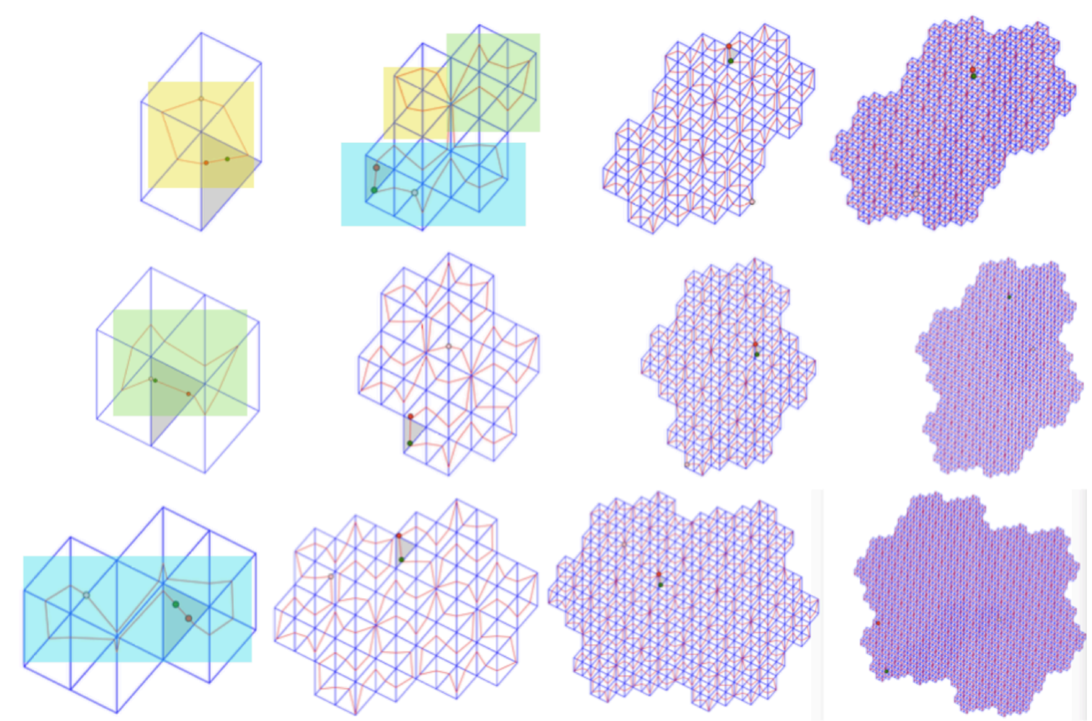
$$\alpha = \frac{\pi}{2}(1 - \boldsymbol{a}), \beta = \frac{\pi}{2}(1 - \boldsymbol{a}^2), \gamma = \frac{\pi}{2}(1 - \boldsymbol{a}^3).$$

$$\boldsymbol{a} + \boldsymbol{a}^2 + \boldsymbol{a}^3 = 1$$

(Davis et al. '17, H.-P.R. '18, **P.-R., '19) :** Inside a Tribonacci billiard, a trajectory ***escapes to infinity*** if and only if it passes through a circumcenter of a tile. It converges to the (famous) ***Rauzy fractal*** and escapes to infinity in a ***non-linear*** way.
Almost always, it passes through ***ALL*** of the tiles.



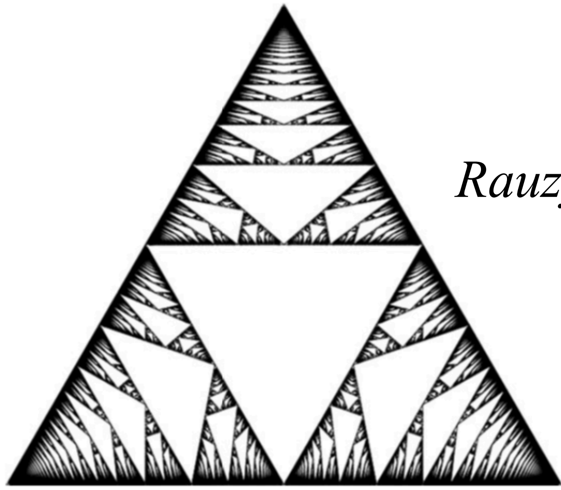




The periods of periodic trajectories in the parallel foliation are doubles of Tribonacci sequence

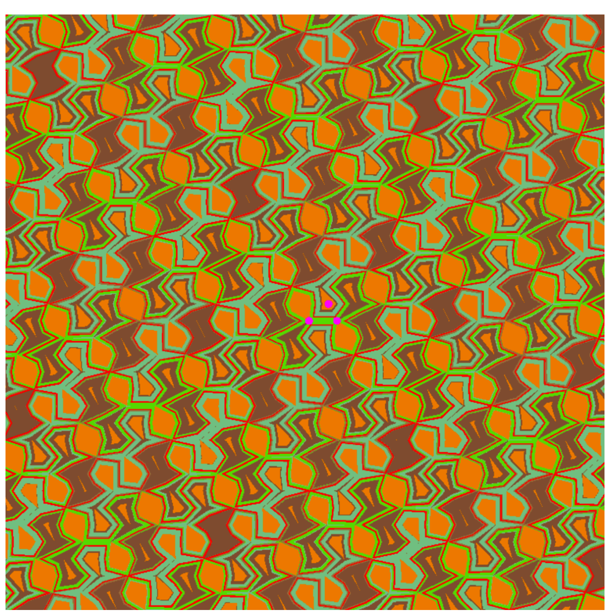
$$1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201, 2209, 4063, 7473, 13745, 25281, 46499, ...$$

Classification of dynamics (**P.-R. 2019**) :

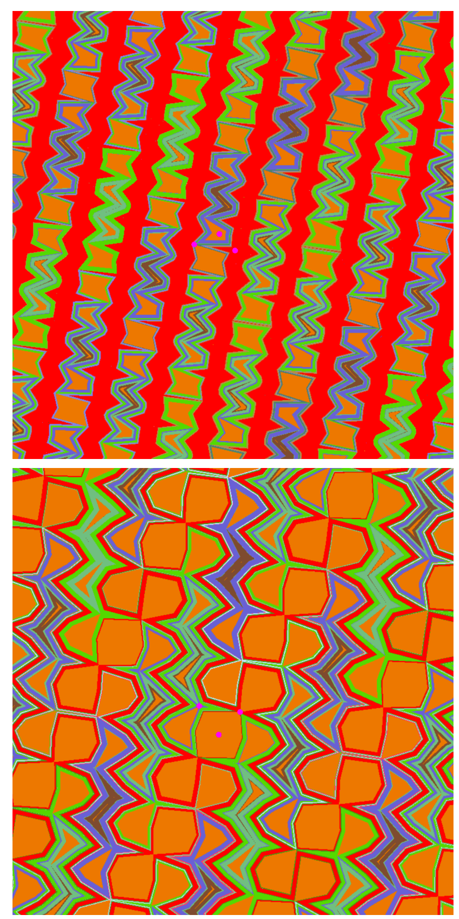


Rauzy gasket

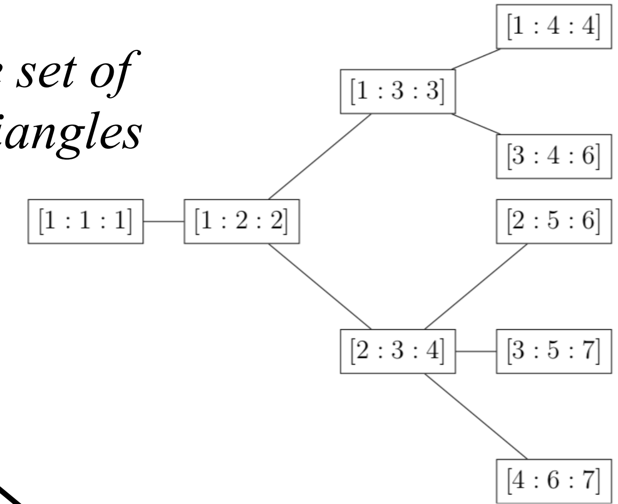
For special parameters of angles in the Rauzy gasket, all trajectories except those passing through circumcenters, are **periodic.**



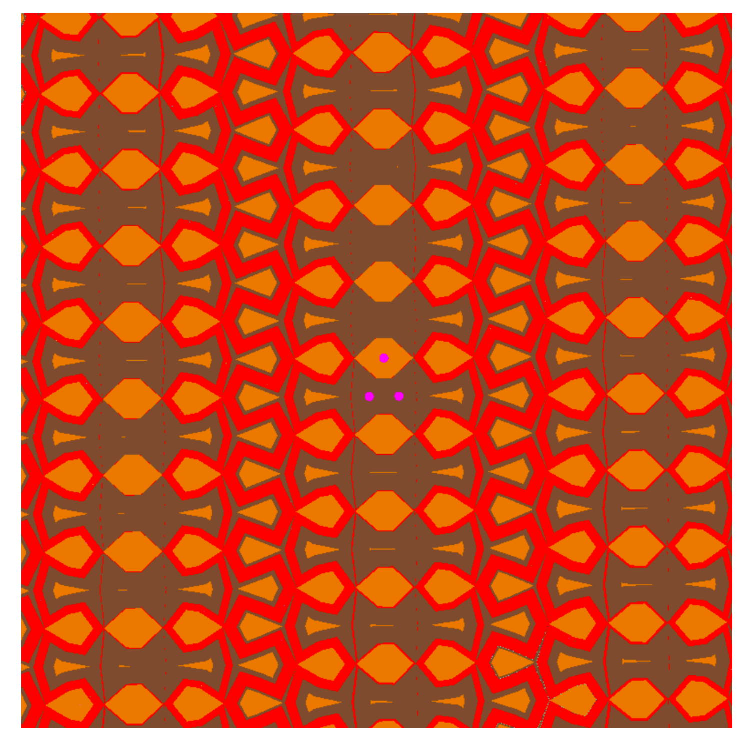
Typical behavior :
On a tiling certain trajectoires are ***periodic*** and certain (closer to the circumcenter) escape linearly.



Countable set of rational triangles

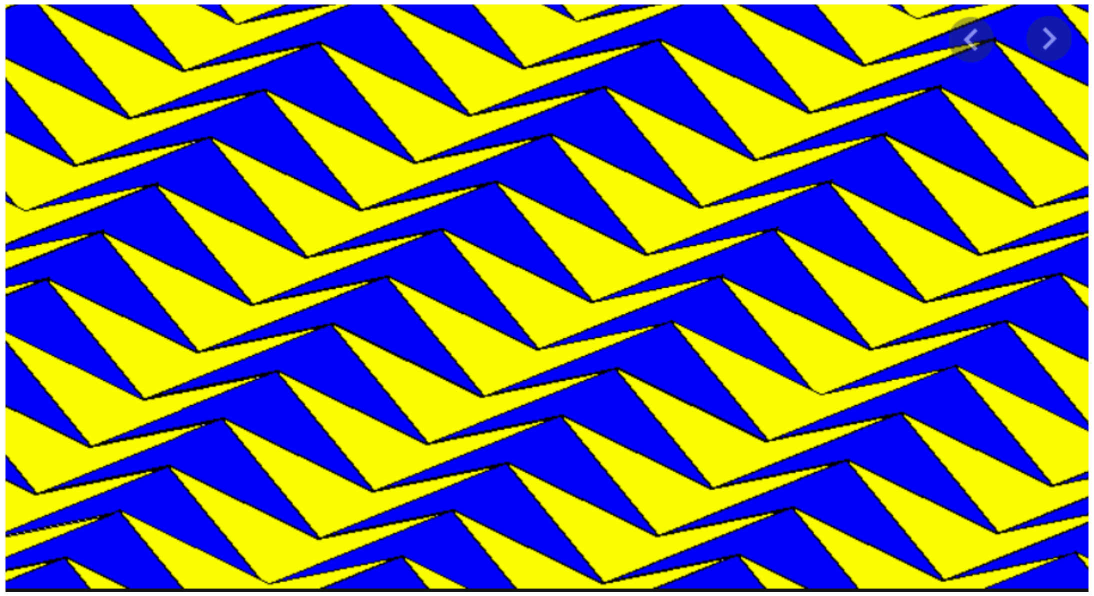


Any trajectory is periodic (as for an equilateral triangle)

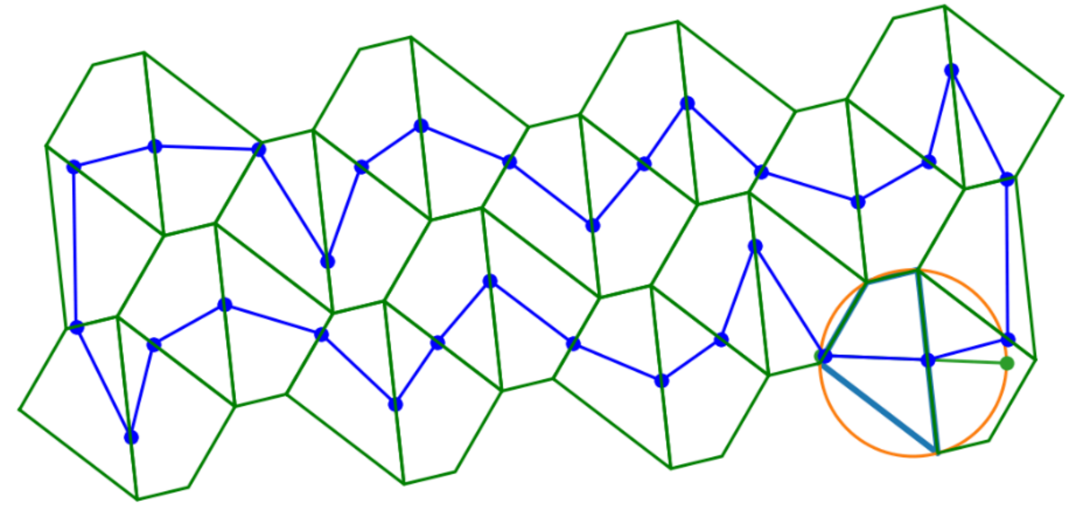


Even though the classification is done, some questions concerning exceptional trajectories are still open !

What next ? Quadrilateral tilings !



every quadrilateral tiles



but only cyclic quadrilateral tilings are foldable!

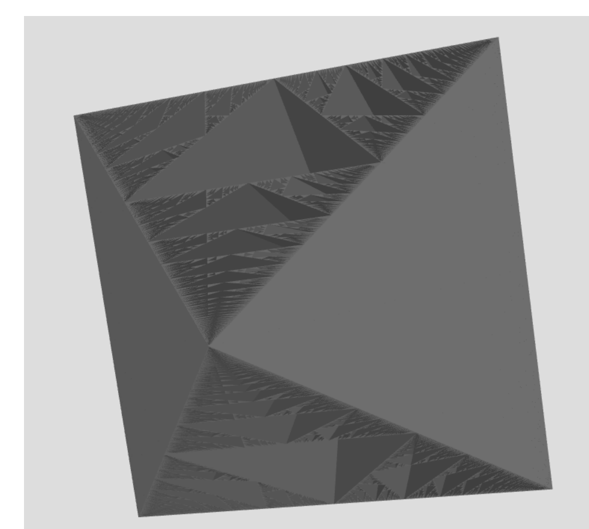
In our work in progress (with Hubert&Mercat and Dynnikov&Skripchenko) we prove that

1. Almost any trajectory on a cyclic quadrilateral tiling is periodic or linearly escaping

2. (the hardest part) Describe explicitly the set of exceptional trajectories (***Novikov's gasket***) and prove that its measure is 0 and Hausdorff dimension is smaller than 3 : for this one needs to work a lot in order to find a *good renormalization*, and then use thermodynamical formalism and Fougeron's criterion to make estimates on Hausdorff dimension (20')

3. This system is equivalent to the Novikov's problem in genus 3 with central symmetry (open for the last 40 years)

Novikov's gasket



What is Novikov's problem ? (1982)

Fix M – a level surface of a smooth function in $\mathbb{T}^3$. Let $\pi : \mathbb{R}^3 \to \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be a standard projection, $\widehat{M} := \pi^{-1}(M)$. Problem : what can one say about the sections of $\widehat{M}$ by a family of parallel planes $H_1x + H_2y + H_3z = \mathrm{const}$, $H = (H_1, H_2, H_3) \in \mathbb{P}^2(\mathbf{R})$?
This defines a foliation $\mathcal{F}$ on M with leaves that are the images of the sections under the projection π.

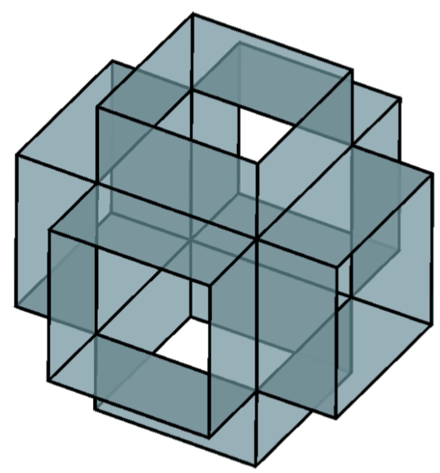
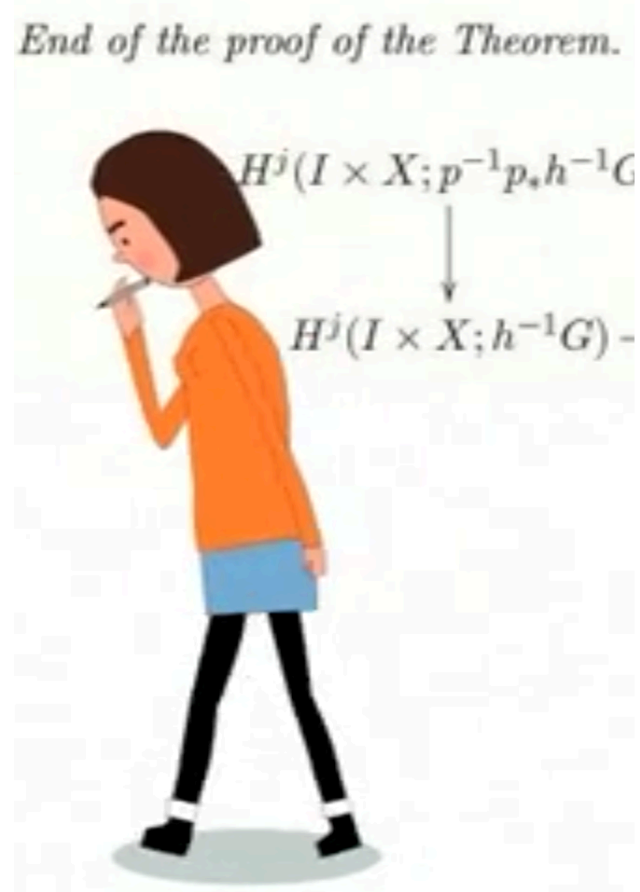


FIGURE – A $\{4, 6|4\}$-polyhedron $M = f^{-1}(1/2)$ for $f(x, y, z) = \mathrm{mid}(|2x - 1|, |2y - 1|, |2z - 1|)$ studied by Dynnikov-De Leo.

THANK YOU FOR YOUR ATTENTION !



If you are interested in tiling billiards, I will be happy to share what I know and work on a joint project !
I am very interested in gathering different perspectives on tiling billiards.

Contact me : olga.romaskevich@math.cnrs.fr
Newsletter (in French) : subscribe on my personal website, olga.pa-ro.net
In my newsletter, I discuss math and art, and share them as I can.