Single recurrence: recent results, open problems, and mysterious examples

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A probability measure preserving system (or MPS) is a triple (X, μ, T) where (X, μ) is a probability measure space and $T : X \to X$ is a transformation preserving μ:

$$\mu(T^{-1}A) = \mu(A) \qquad \text{for all measurable } A \subset X$$

In this talk we only consider invertible systems (T is invertible).

A topological dynamical system (X, T) is a compact metric space X together with a continuous map $T : X \to X$.

We say (X, T) is minimal if the only nonempty closed T-invariant subset of X is X.

Equivalently, (X, T) is minimal if for every nonempty open $U \subset X$, we have $\bigcup_{n=1}^{m} T^{-n}U = X$ for some $m \in \mathbb{N}$.

Group rotations

A group rotation (K, R_α) is a topological system where K is a compact abelian group, $\alpha \in K$, and $R_\alpha x := x + \alpha$.

Writing m for Haar probability measure on K, we get a MPS (K, m, R_α).

Example: $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Identifying $\mathbb{T}$ with the unit interval $[0, 1)$, fix an $\alpha \in \mathbb{T}$. Then (K, R_α) is a group rotation.

The Poincaré Recurrence Theorem

Theorem (PRT)

If (X, μ, T) *is a MPS and* $\mu(A) > 0$*, there exists* $n \in \mathbb{N}$ *such that* $\mu(A \cap T^{-n}A) > 0$*.*

We'll give two proofs.

Definition

$S \subset \mathbb{Z}$ is a set of measurable recurrence if $\forall$ MPS (X, μ, T) and all $A \subset X$ with $\mu(A) > 0$, we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in S$.

PRT says that $\mathbb{N}$ is a set of measurable recurrence.

Definition

$E \subset \mathbb{Z}$ is a set of measurable recurrence if $\forall$ MPS (X, μ, T) and all $A \subset X$ with $\mu(A) > 0$, we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in E$.

Examples:

- Sets of the form $(S - S) \cap \mathbb{N}$, where S is infinite. $S - S$ means $\{s - s' : s, s' \in S\}$.
- Sets containing arbitrarily long intervals: $\bigcup_{n=1}^{\infty} [2^n, 2^n + n]$.
- $\{n^2 : n \in \mathbb{N}\}$ (Furstenberg [Fur77], Sárkőzy [Ś78]).
- $S_{5/2} := \{\lfloor n^{5/2} \rfloor : n \in \mathbb{N}\}$.
- Every translate of $S_{5/2}$ is a set of measurable recurrence.

Non-examples:

- $\{n^2 + 1 : n \in \mathbb{N}\}$ is not a set of measurable recurrence.
- The set $\{n! : n \in \mathbb{N}\}$ is not a set of measurable recurrence.
- Lacunary sets: if $S = \{s_1 < s_2 < \dots\}$ and $\inf s_{n+1}/s_n > 1$, S is not a set of measurable recurrence.

Theorem (PRT)

If $S \subset \mathbb{Z}$ is infinite, then $(S - S) \cap \mathbb{N}$ is a set of mble recurrence.

Proof.

Fix infinite $S \subset \mathbb{Z}$, an MPS (X, μ, T), and $A \subset X$ with $\mu(A) > 0$.

We will find $s \neq s' \in S$ such that $\mu(A \cap T^{s-s'}A) > 0$.

Let $k \in \mathbb{N}, k > \frac{1}{\mu(A)}$, and let $s_1, \dots, s_k$ be distinct elements of S.

Each $T^{-s_j}A$ has measure $\mu(A)$, so $\sum_{j=1}^{k} \mu(T^{-s_j}A) = k\mu(A) > 1$.

Since $\mu(X) = 1$, at least two of the sets must intersect with positive measure, meaning

$$\mu(T^{-s_i}A \cap T^{-s_j}A) > 0$$

Since T preserves μ, we have

$$\mu(A \cap T^{s_i - s_j}A) = \mu(T^{-s_i}A \cap T^{-s_j}A) > 0. \qquad \square$$

Theorem (Mean Ergodic Theorem)

Let (X, μ, T) be a MPS and $f \in L^2(\mu)$. If $I_k = \{n_k, \ldots, m_k\}$ are intervals in $\mathbb{Z}$ with $|I_k| \to \infty$, then

$$\lim_{k\to\infty} \frac{1}{|I_k|} \sum_{n\in I_k} f \circ T^n = P_{inv} f \qquad \text{in } L^2(\mu)$$

where P_{inv} is orth. proj. onto the space of T-invariant functions.

Corollary (Khintchine)

If $A \subset X$, then $\lim_{k\to\infty} \frac{1}{|I_k|} \sum_{n\in I_k} \mu(A \cap T^{-n}A) \geq \mu(A)^2$.
Consequently: for all $\varepsilon > 0$ there exists $n \in \bigcup_{k\in\mathbb{N}} I_k$ such that

$$\mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon.$$

Khintchine: $\lim_{k\to\infty} \frac{1}{|I_k|}\sum_{n\in I_k} \mu(A\cap T^{-n}A) \geq \mu(A)^2$.

Proof.

Let $f = 1_A$, so $\mu(A\cap T^{-n}A) = \int f\cdot f\circ T^n\, d\mu$.

$$
\begin{aligned}
\frac{1}{|I_k|}\sum_{n\in I_k}\mu(A\cap T^{-n}A) &= \frac{1}{|I_k|}\sum_{n\in I_k}\int f\cdot f\circ T^n\, d\mu \\
&= \int f\cdot \frac{1}{|I_k|}\sum_{n\in I_k} f\circ T^n\, d\mu \\
&\underset{k\to\infty}{\longrightarrow} \int f\cdot P_{inv}f\, d\mu && \text{mean erg. thm.} \\
&= \int P_{inv}f\cdot P_{inv}f\, d\mu && \text{orth. proj.} \\
&\geq \left(\int P_{inv}f\, d\mu\right)^2 && \text{C-S, or Jensen} \\
&= \mu(A)^2. && \square
\end{aligned}
$$

A sequence of integers s_n is Hartman uniformly distributed if

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp(2\pi i s_n t) = 0 \qquad \text{for all } t \in (0,1)$$

Examples:

- $s_n = n$
- $s_n = \lfloor n^{5/2} \rfloor$

See [Nai98], [BKQW05], for more.

Theorem

If (s_n) is Hartman-u.d., (X, μ, T) is a MPS, and $f \in L^2(\mu)$, then

$$(2) \qquad \frac{1}{N}\sum_{n=1}^{N} f \circ T^{s_n} \underset{N\to\infty}{\longrightarrow} P_{inv} f.$$

Consequently, $\{s_n : n \in \mathbb{N}\}$ is a set of measurable recurrence.

(1) says s_n has the correct ergodic averages for torus rotations,
(2) says s_n has the correct ergodic averages for every MPS.

Definition

Let $S \subset \mathbb{Z}$. We say that S is a set of

1. measurable recurrence if $\forall$ (X, μ, T), $A \subset X$ with $\mu(A) > 0$, there exists $n \in S$ such that $\mu(A \cap T^{-n}A) > 0$.
2. topological recurrence if $\forall$ minimal (X, T), nonempty open $U \subset X$ there exists $n \in S$ such that $U \cap T^{-n}U \neq \varnothing$.
3. Bohr recurrence if for $\forall$ minimal group rotations (K, R_α), non-$\varnothing$ open $U \subset K$, $\exists$ $n \in S$ such that $U \cap R_\alpha^{-n}U \neq \varnothing$.

For $x \in \mathbb{R}$, let $\|x\| :=$ distance from x to the nearest integer. For $(x_1, \ldots, x_d) \in \mathbb{R}^d$, let $\|x\| = \max_{j \leq d} \|x_j\|$.

S is a set of Bohr recurrence iff for all $d \in \mathbb{N}$ and all d-tuples $\alpha = (\alpha_1, \ldots, \alpha_d)$ of real numbers, we have

$$\inf_{n \in S} \|n\alpha\| = 0.$$

Katznelson's and Veech's problem

The definitions easily yield:
S is mble rec. $\implies$ S is top. rec. $\implies$ S is Bohr rec.

S is a set of top. rec. $\not\Rightarrow$ S is a set of mble rec.
(Kriz, [Kri87]), answering a question of Bergelson.

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

The natural intuition is no: topological systems can be much more complicated than group rotations.

But recall: if (s_n) has correct ergodic averages for torus rotations, then it has correct ergodic averages for arbitrary MPSs.

On the following slides we provide some evidence that every set of Bohr recurrence is a set of topological recurrence.

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

Let $S = \{2^n : n \in \mathbb{N}\}$.

We will prove that if $E \subset S - S$ and E is a set of Bohr recurrence, then E is a set of topological recurrence.

This is not vacuous: $S - S$ is a set of measurable recurrence, by Poincaré's recurrence theorem.

The conclusion cannot be improved to "then E is a set of measurable recurrence," by Kriz's construction ([Kri87], implicitly).

Inspired by [KR99] on lacunary sets and the Bohr topology.

Lemma

Let $S \subset \mathbb{Z}$. The following are equivalent.

(i) *S is a set of topological recurrence.*

(ii) *For all $k \in \mathbb{N}$ and every $f : \mathbb{Z} \to \{1, \ldots, k\}$, there exists $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $f(b) = f(a)$.*

Proof of (ii) $\implies$ (i).

Let (X, T) be minimal, $U \subset X$ nonempty and open.

Fix $k \in \mathbb{N}$ such that $X = \bigcup_{m=1}^{k} T^{-m} U$(minimality).

Fix $x \in X$, define $f(n) = \min\{m : T^n x \in T^{-m} U\}$, so $f(n) \leq k$.

By (ii), choose $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $f(b) = f(a)$.

Writing m for the common value of $f(b)$, $f(a)$, we have

$$\begin{aligned} & T^b x \in T^{-m} U \quad \text{and} \quad T^a x \in T^{-m} U \\ \implies & \; T^m x \in T^{-b} U \cap T^{-a} U \\ \implies & \quad U \cap T^{b-a} U \neq \varnothing. \end{aligned}$$

□

Lemma

Let $S \subset \mathbb{Z}$. The following are equivalent:

(i) *S is a set of Bohr recurrence.*

(ii) *For all trigonometric polynomials $p : \mathbb{Z} \to \mathbb{C}$, $\varepsilon > 0$, there exists $m \in S$ such that $|p(n+m) - p(n)| < \varepsilon$ for all $n \in \mathbb{Z}$.*

Proof of (i) $\implies$ (ii).

Fix $p(n) := \sum_{j=1}^{d} c_j e(n\alpha_j)$, $\varepsilon > 0$. Let $C = 1 + \sum |c_j|$.

Since S is a set of Bohr recurrence, we can find $m \in S$ such that $\|m\alpha_j\| < \varepsilon/4C$ for each j, meaning $|e(m\alpha_j) - 1| < \varepsilon/C$. Then

$$\begin{aligned} |p(n+m) - p(n)| &\leq \sum |c_j||e((n+m)\alpha_j) - e(n\alpha_j)| \\ &= \sum |c_j||e(m\alpha_j) - 1| \\ &< \sum |c_j|\varepsilon/C \\ &< \varepsilon \end{aligned}$$

□

I_0 sets

We say $S \subset \mathbb{Z}$ is an I_0-set if for all bounded $f : S \to \mathbb{C}$ and $\varepsilon > 0$ there is a trigonometric polynomial p such that $|f(s) - p(s)| < \varepsilon$ for all $s \in S$.

We say $S = \{s_1 < s_2 < s_3 < \dots\}$ is lacunary if $\inf s_{n+1}/s_n > 1$.

Theorem (Strzelecki [Str63])

If $S \subset \mathbb{N}$ is lacunary, then S is an I_0 set.

For example, $\{3^n : n \in \mathbb{N}\}$ is an I_0 set.

cf. [Le20], [KR99].

Lemma (Non-separation in differences of I_0 sets)

Let $S \subset \mathbb{Z}$ be an I_0 set. If $E \subset S - S$ and E is a set of Bohr recurrence, then E is a set of topological recurrence.

We recall the two preceding lemmas.

Lemma

Let $E \subset \mathbb{Z}$. The following are equivalent.

(i) *E is a set of topological recurrence.*

(ii) *For all $k \in \mathbb{N}$ and every $f : \mathbb{Z} \to \{1, \dots, k\}$, there exists $a, b \in \mathbb{Z}$ such that $b - a \in E$ and $f(b) = f(a)$.*

The following are equivalent:

(iii) *E is a set of Bohr recurrence.*

(iv) *For all trigonometric polynomials $p : \mathbb{Z} \to \mathbb{C}$, $\varepsilon > 0$, there exists $m \in E$ such that $|p(n + m) - p(n)| < \varepsilon$ for all $n \in \mathbb{Z}$.*

Lemma

Let $S \subset \mathbb{Z}$ be an I_0 set. If $E \subset S - S$ and E is a set of Bohr recurrence, then E is a set of topological recurrence.

Proof.

Let $E \subset S - S$ be Bohr recurrent. To prove E is topologically recurrent, we fix $k \in \mathbb{N}$ and an arbitrary $f : \mathbb{Z} \to \{1, \ldots, k\}$, and will prove $\exists\, a, b \in \mathbb{Z}$ such that $f(b) = f(a)$ and $b - a \in E$.

S is I_0, so let p be a trig. poly. with $|f(s) - p(s)| < \frac{1}{3}\ \forall s \in S$.

E is Bohr recurrent, so fix $m \in E$ with $|p(n+m) - p(n)| < \frac{1}{3} \forall n$.

Since $E \subset S - S$, write $m = b - a$, where $a, b \in S$. In particular $|p(b) - p(a)| = |p(a+m) - p(a)| < \frac{1}{3}$, so

$$\begin{aligned} |f(b) - f(a)| &= |f(b) - p(b) + p(b) - p(a) + p(a) - f(a)| \\ &\leq |f(b) - p(b)| + |p(b) - p(a)| + |p(a) - f(a)| < 1, \end{aligned}$$

so $f(b) = f(a)$. Since $b - a \in E$, we are done. □

Separating recurrence properties

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

If there is a set of Bohr recurrence which is not a set of topological recurrence, it cannot be a subset of a difference set of an I_0 set.

Lemma (Non-separation)

If $S \subset \mathbb{Z}$ is an I_0 set and $E \subset S - S$ is a set of Bohr recurrence, then E is a set of topological recurrence.

Question

Can the hypothesis "S is an I_0 set" be weakened?

Theorem (Kriz, [Kri87])

If $S \subset \mathbb{Z}$ is infinite, then there is a set $E \subset S - S$ which is a set of topological recurrence but not of measurable recurrence.

Conjecture

If $S \subset \mathbb{Z}$ is a set of measurable recurrence, then there is a set $S' \subset S$ such that S' is a set of topological recurrence and not a set of measurable recurrence.

No such result is known for "local" recurrence properties. For some translation-invariant recurrence properties, separation is known.

Theorem ([Gri20])

If every translate of S is a set of measurable recurrence, $\exists S' \subset S$ such that every translate of S' is a set of Bohr recurrence and S' is not a set of measurable recurrence.

Conjecture

If every translate of S is a set of measurable recurrence, then $\exists\, S' \subset S$ such that every translate of S' is a set of topological recurrence and not a set of measurable recurrence.

Sets with unknown recurrence properties

The following are known to be sets of Bohr recurrence, but not known to be sets of topological recurrence.

1. $\{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$
   B. N. Givens PhD thesis [Giv03], Givens and Kunen [GK03].
2. Grivaux and Roginskaya's examples [GR13]
3. Translates of the IP set generated by the Erdős-Taylor sequence [Kat73], [GM79]
4. $\{n!2^m3^k : n, m, k \in \mathbb{N}\}$ (said to be Bohr recurrent in [FM12])
5. The sets constructed in [Gri20]: S is dense in the Bohr topology, not a set of measurable recurrence.

The first four could possibly be sets of measurable recurrence.

Lemma ([GK03], Lemma 3.3, cf [Giv03])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$ *is a set of Bohr recurrence.*

Proof.

Fix $d \in \mathbb{N}$ and $\alpha \in \mathbb{T}^d$. We must find $m \in S$ such that $\|m\alpha\| < \varepsilon$. Cover $\mathbb{T}^d$ with sets $U_1, \dots, U_k$ of diameter $< \varepsilon/2$. For $n \in \mathbb{N}$, let

$$f(n) := \min\{j : 7^n\alpha \in U_j\}$$

This is a finite coloring of $\mathbb{N}$.
By van der Waerden's theorem on arithmetic progressions, there are $n, d \in \mathbb{N}$, $j \leq k$ so that $7^n\alpha, 7^{n+d}\alpha, 7^{n+2d}\alpha$ all lie in U_j. Thus

$$\|7^{n+d}\alpha - 7^n\alpha\| < \varepsilon/2 \quad \text{and} \quad \|7^{n+2d}\alpha - 7^n\alpha\| < \varepsilon/2,$$

and the triangle inequality yields

$$\|(7^{n+2d} + 7^{n+d} - 2 \cdot 7^n)\alpha\| < \varepsilon.$$

□

We really proved:

Lemma ([GK03], Lemma 3.3, cf [Giv03])

Let (a_n) be a sequence of integers. Then

$$S := \{a_{n+2d} + a_{n+d} - 2a_n : n, d \in \mathbb{N}\}$$

is a set of Bohr recurrence.

Sparser sequences, like $a_n = 10^{10^n}$, are more interesting here.

Question

Is every such S a set of topological recurrence? A set of measurable recurrence?

Such sets arise in the study of Bohr topologies of certain abelian groups, like the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$: [Kun98], [Dik01], [DW01], [dLD08].

Grivaux and Roginskaya's examples

The following two facts suggest an appealing strategy for constructing sets of Bohr recurrence lacking stronger recurrence properties.

Recurrence properties of long arithmetic progressions $\{a, a+d, \ldots, a+(k-1)d\}$ can be understood using diophantine approximation arguments.

Lacunary sets are not Bohr recurrent.

[GR13] inductively constructs a set S as a union of increasingly sparse arithmetic progressions and uses diophantine approximation arguments to prove it is a set of Bohr recurrence.

There are several parameters in the construction. Can some choice of those parameters produce a set of Bohr recurrence which is not a set of topological recurrence?

The IP set generated by the Erdős-Taylor sequence

The Erdős-Taylor sequence is defined by $n_1 = 1$ and $n_{k+1} = kn_k + 1$.

Let S be the IP set generated by $(n_k)_{k\in\mathbb{N}}$:

$$S := \{n_{i_1} + \cdots + n_{i_r} : i_1 < \cdots < i_r, r \in \mathbb{N}\}$$

Katznelson proved that S is dense in the Bohr topology, but is not Hartman-u.d. [Kat73], [GM79]. Used to construct rigidity sequences in [BGM19], [Gri13].

In particular, the translate $S + 1$ is a set of Bohr recurrence.

Is $S + 1$ a set of topological recurrence? Measurable recurrence? Strong recurrence?

What about other translates?

The set $\{n!2^m3^k : n, m, k \in \mathbb{N}\}$

$$S := \{n!2^m3^k : n, m, k \in \mathbb{N}\}$$

[FM12] asks: what are its recurrence properties?

No explicit proof or disproof of Bohr recurrence, topological recurrence, or measurable recurrence appears in print.

Hamming Balls and Bohr-Hamming Balls

[Gri20] constructs sets $S \subset \mathbb{Z}$ which are dense in the Bohr topology and are not sets of measurable recurrence.

It builds on the standard method for constructing dense sets A where $A - A$ lacks structure (niveau sets, Kriz's example)

Question

Are the sets constructed in [Gri20] also sets of topological recurrence?

Is the Kriz example the only way to separate toplogical recurrence from measurable recurrence?

[Kri87] constructs a set of topological recurrence which is not a set of measurable recurrence.

Topological recurrence is obtained by locating Kneser graphs of high chromatic number (Lovász's theorem) in a Cayley graph associated to a set $S \subset \mathbb{Z}$.

This was simplified by Ruzsa [McC99], [McC95], [Wei00], and extended: [For90], [Gri20].

Variations on Kriz's construction are the only known examples of dense sets A where $A - A$ lacks some prescribed structure.

Question

Is there a fundamentally different construction of a set of topological recurrence which is not a set of measurable recurrence?

References I

Catalin Badea, Sophie Grivaux, and Etienne Matheron, *Rigidity sequences, kazhdan sets and group topologies on the integers*, 2019, arxiv:1812.09014.

Michael Boshernitzan, Grigori Kolesnik, Anthony Quas, and Máté Wierdl, *Ergodic averaging sequences*, J. Anal. Math. **95** (2005), 63–103. MR 2145587

Dikran Dikranjan, *Continuous maps in the Bohr topology*, Appl. Gen. Topol. **2** (2001), no. 2, 237–270. MR 1890040

Lorenzo de Leo and Dikran Dikranjan, *Straightening theorem for bounded abelian groups*, Topology Appl. **155** (2008), no. 17-18, 2158–2176. MR 2458001

References II

- Dikran Dikranjan and Stephen Watson, *A solution to van Douwen's problem on Bohr topologies*, J. Pure Appl. Algebra **163** (2001), no. 2, 147–158. MR 1846658
- Nikos Frantzikinakis and Randall McCutcheon, *Ergodic theory: recurrence*, Mathematics of complexity and dynamical systems. Vols. 1–3, Springer, New York, 2012, pp. 357–368. MR 3220681
- Alan Hunter Forrest, *Recurrence in dynamical systems: A combinatorial approach*, ProQuest LLC, Ann Arbor, MI, 1990, Thesis (Ph.D.)–The Ohio State University. MR 2685439
- Harry Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Analyse Math. **31** (1977), 204–256. MR 498471

Berit Nilsen Givens, *Hypergraphs and chromatic numbers, with applications to the Bohr topology*, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)–The University of Wisconsin - Madison. MR 2704616

Berit Nilsen Givens and Kenneth Kunen, *Chromatic numbers and Bohr topologies*, Topology Appl. **131** (2003), no. 2, 189–202. MR 1981873

Colin C. Graham and O. Carruth McGehee, *Essays in commutative harmonic analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 238, Springer-Verlag, New York-Berlin, 1979. MR 550606

- Sophie Grivaux and Maria Roginskaya, *Some new examples of recurrence and non-recurrence sets for products of rotations on the unit circle*, Czechoslovak Math. J. **63(138)** (2013), no. 3, 603–627. MR 3125645
- Sophie Grivaux, *IP-Dirichlet measures and IP-rigid dynamical systems: an approach via generalized Riesz products*, Studia Math. **215** (2013), no. 3, 237–259. MR 3080781
- John T. Griesmer, *Separating bohr denseness from measurable recurrence*, 2020, arxiv:2002.06994.
- Y. Katznelson, *Sequence of integers dense in the bohr group*, Proc. Royal. Inst. Techn. Stockholm (1973), 79–86.
- ______, *Chromatic numbers of Cayley graphs on $\mathbb{Z}$ and recurrence*, vol. 21, 2001, Paul Erdős and his mathematics (Budapest, 1999), pp. 211–219. MR 1832446

- Kenneth Kunen and Walter Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 1, 117–137. MR 1681658
- Igor Kriz, *Large independent sets in shift-invariant graphs: solution of Bergelson's problem*, Graphs Combin. **3** (1987), no. 2, 145–158. MR 932131
- Kenneth Kunen, *Bohr topologies and partition theorems for vector spaces*, Topology Appl. **90** (1998), no. 1-3, 97–107. MR 1648302
- Anh N. Le, *Interpolation sets and nilsequences*, Colloq. Math. **162** (2020), no. 2, 181–199. MR 4128459

References VI

Randall McCutcheon, *Three results in recurrence*, Ergodic theory and its connections with harmonic analysis (Alexandria, 1993), London Math. Soc. Lecture Note Ser., vol. 205, Cambridge Univ. Press, Cambridge, 1995, pp. 349–358. MR 1325710

______, *Elemental methods in ergodic Ramsey theory*, Lecture Notes in Mathematics, vol. 1722, Springer-Verlag, Berlin, 1999. MR 1738544

R. Nair, *On uniformly distributed sequences of integers and Poincaré recurrence*, Indag. Math. (N.S.) **9** (1998), no. 1, 55–63. MR 1618231

A. Sárkőzy, *On difference sets of sequences of integers. I*, Acta Math. Acad. Sci. Hungar. **31** (1978), no. 1-2, 125–149. MR 466059

- E. Strzelecki, *On a problem of interpolation by periodic and almost periodic functions*, Colloq. Math. **11** (1963), 91–99. MR 160080
- William A. Veech, *The equicontinuous structure relation for minimal Abelian transformation groups*, Amer. J. Math. **90** (1968), 723–732. MR 232377
- Benjamin Weiss, *Single orbit dynamics*, CBMS Regional Conference Series in Mathematics, vol. 95, American Mathematical Society, Providence, RI, 2000. MR 1727510