Matchings in Graphs

Let $G = (V, E)$ be a (finite, simple) graph. For $X \subseteq V$, its *neighbourhood* is $\Gamma(X) := \{y \in V : \exists\, x \in X\ xy \in E\}$. A *matching* is a subset M of E consisting of disjoint edges. Let $V(M) = \cup_{\{x,y\}\in M}\{x, y\}$ denote the set of vertices covered by it. Call M *perfect* if $V(M) = V$, that is, if every vertex of G is covered by an edge of M.

Hall's Marriage Theorem: a bipartite graph G has a matching covering every vertex of A if and only if $|\Gamma(X)| \geq |X|$ for every $X \subseteq A$.

König-Egerváry Theorem: The maximum size of a matching in a bipartite graph G is equal to the minimum size of a subset $X \subseteq V$ such that every edge intersects X.

König Theorem: Let $\Delta := \max\{|\Gamma(\{x\})| : x \in V\}$ be the maximum degree of G. If G is bipartite, then

1. there is a matching M which covers every vertex of degree Δ;
2. one can colour E with Δ colours so that no two adjacent edges have the same colour.

A sufficient and necessary condition for the existence of a perfect matching in arbitrary graphs is given by the *Tutte 1-Factor Theorem* (but is appears rarely in problem-solving).

A great book of problems in combinatorics (with hints and detailed solutions) is L.Lovász *"Combinatorial Problems and Exercises"*.

Problem 1 Let H be a finite group and let K be a subgroup of H. Show that there exist elements $h_1, h_2, \ldots, h_n \in H$ with $n = [H : K]$, such that $h_1K, h_2K, ..., h_nK$ are the left cosets of K and $Kh_1, Kh_2, \ldots, Kh_n$ are the right cosets of K.

Problem 2 Let M be a matching in a (not necessarily biparite) graph $G = (V, E)$. Recall that an *M-augmenting path* is a path that stars and end with unmatched vertices and whose edges alternate between $E \setminus M$ and M. Let $k \in \mathbb{N}$. Show that if there is no M-augmenting path with most $2k - 1$ edges, then the maximum size of a matching in G is at most $|M| + \min\{\frac{1}{2k+1}|V|, \frac{1}{k}|M|\}$.

Problem 3 Let G be a bipartite graph with parts $V_1 \cup V_2$ and let $M_1, M_2 \subseteq E(G)$ be two matchings. Prove that there is a matching M in G such that

$$V(M) \supseteq (V_1 \cap V(M_1)) \cup (V_2 \cap V(M_2)).$$

Problem 4 (a) Let G be a bipartite graph with parts $V_1 \cup V_2$ (possibly infinite) such that G is *locally finite* (that is, every vertex is incident to finitely many edges). Assuming the Axiom of Choice, prove that G has a perfect matching if and only if for every $i = 1, 2$ and every finite $X \subseteq V_i$ we have $|\Gamma(X)| \geq |X|$.

(b) Show that the local finiteness assumption cannot be dropped in (a).