Some problems about matrices

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1 Minkowski's inequality for determinants

Lemma 1.1. *[Vil03, Lemma 5.23] Let A and B be two non-negative symmetric $n \times n$ matrices, and $\lambda \in [0,1]$. Assume that A is invertible. Then*

$$\det(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda(\det A)^{1/n} + (1-\lambda)(\det B)^{1/n}, \tag{1}$$

and

$$\det(\lambda A + (1-\lambda)B) \geq (\det A)^{\lambda}(\det B)^{1-\lambda}. \tag{2}$$

Proof. We first prove (1). Since $\det(\lambda A) = \lambda^n \det A$, to prove (1) it is sufficient to prove that

$$\det(A+B)^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}. \tag{3}$$

This inequality is known as Minkowski's inequality. Since $A+B = A^{1/2}(I+A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}(I+C)A^{1/2}$, where $C = A^{-1/2}BA^{-1/2}$ is a non-negative symmetric matrix, and $\det(MN) = (\det M)(\det N)$ we have

$$\det(A+B)^{1/n} = (\det A)^{1/n}\det(I+C)^{1/n}, \quad (\det A)^{1/n} + (\det B)^{1/n} = (\det A)^{1/n}\Big[1+(\det C)^{1/n}\Big].$$

We next show that

$$\det(I+C)^{1/n} \geq 1 + (\det C)^{1/n} \quad \text{for all } C \text{ non-negative and symmetric,}$$

which will imply (3). Since C is non-negative symmetric, it has nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$ and

$$\det(I+C)^{1/n} = \prod_{i=1}^{n}(1+\lambda_i)^{1/n} \quad \text{and} \quad (\det C)^{1/n} = \prod_{i=1}^{n}\lambda_i^{1/n}.$$

We need to prove

$$\prod_{i=1}^{n}(1+\lambda_i)^{1/n} \geq 1 + \prod_{i=1}^{n}\lambda_i^{1/n}.$$

This inequality indeed holds true since using the Arithmetic-Geometric inequality we have

$$\frac{1+\prod_{i=1}^{n}\lambda_i^{1/n}}{\prod_{i=1}^{n}(1+\lambda_i)^{1/n}} = \prod\Big(\frac{1}{1+\lambda_i}\Big)^{1/n} + \prod\Big(\frac{\lambda_i}{1+\lambda_i}\Big)^{1/n} \leq \frac{1}{n}\sum\frac{1}{1+\lambda_i} + \frac{1}{n}\sum\frac{\lambda_i}{1+\lambda_i} = 1.$$

We now prove (2). From (1) and the Arithmetic-Geometric inequality we have

$$\det(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda(\det A)^{1/n} + (1-\lambda)(\det B)^{1/n} \geq (\det A)^{\lambda/n}(\det B)^{(1-\lambda)/n},$$

which implies that

$$\det(\lambda A + (1-\lambda)B) \geq (\det A)^{\lambda}(\det B)^{(1-\lambda)},$$

that is (2) as expected. □

2 Hadamard's Inequality

Theorem 2.1 (Hadamard's inequality)**.**

$$|\det A| \leq \prod_{i=1}^{n}\|\mathbf{a}_i\|, \tag{4}$$

where $\{\mathbf{a}_i\}_{i=1}^n$ *are (real vectors) columns of* A *and* $\|\cdot\|$ *is the Euclidean norm.*

Proof. The inequality is obviously true If A is singular. Therefore, assume that the column of A are linearly independent. By dividing each column by its length, the inequality is equivalent to the special case where each column has length 1. Suppose that $\{\mathbf{b}_i\}_{i=1}^n$ are unit column vectors and B has the $\{\mathbf{b}_i\}$ as column. We need to show that

$$|\det B| \leq 1.$$

Indeed, let $C = B^T B$. Then C is non-negative symmetric whose diagonal entries are all 1. Thus $\mathrm{tr}C = n$. Let $\lambda_1, \dots, \lambda_n \geq 0$ be the eigenvalues of C. By the arithmetic-geometric inequality we have

$$(\det B)^2 = \det C = \prod_{i=1}^{n}\lambda_i \leq \Big(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\Big)^n = \Big(\frac{1}{n}\mathrm{tr}C\Big)^n = 1,$$

i.e., $|\det B| \leq 1$ as required. □

Corollary 2.2. *Let A be a $n \times n$ positive definite matrix. Then*

$$|\det A| \leq \prod_{i=1}^{n} A_{ii}. \tag{5}$$

Proof. Sine A is positive definite, there exists B such that $A = B^T B$. Let $\mathbf{b}_i$ are the columns of B. We have

$$\det A = (\det B)^2 \leq \prod \|\mathbf{b}_i\|^2 = \prod A_{ii}. \tag{6}$$

□

Proposition 2.3. *[Dan01, Theorem 2.8] Let A and B be $n \times n$ positive definite matrices. Then*

$$n(\det A \cdot \det B)^{\frac{m}{n}} \leq \operatorname{tr}(A^m B^m) \tag{7}$$

for any positive integer m.

Proof. Let $\lambda_1, \ldots, \lambda_n > 0$ be eigenvalues of A and suppose that $A = P^T \Lambda P$ where P is an orthonormal matrix and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Let $b_{11}(m), \ldots, b_{nn}(m)$ denote the diagonal elements of $(P^T BP)^m$. Since the trace operator is invariant under permutation, we have

$$\begin{aligned}
\frac{1}{n}\operatorname{tr}(A^m B^m) &= \frac{1}{n}\operatorname{tr}(P^T \Lambda^m P B^m) \\
&= \frac{1}{n}\operatorname{tr}(\Lambda^m P^T B^m P) \\
&= \frac{1}{n}\operatorname{tr}(\Lambda^m (P^T BP)^m) \\
&= \frac{1}{n}\sum_{i=1}^{n} \lambda_i^m b_{ii}(m).
\end{aligned}$$

Using the last identity and the arithmetic-geometric inequality, we have

$$\frac{1}{n}\operatorname{tr}(A^m B^m) \geq \prod (\lambda_i)^{\frac{m}{n}} \prod (b_{ii}(m))^{\frac{1}{n}} \tag{8}$$

On the other hand from (5) we have

$$(\det A \cdot \det B)^{\frac{m}{n}} = (\det \Lambda^m \det P^T B^m P)^{\frac{1}{n}} \leq \prod \prod \lambda_i^{\frac{m}{n}} \prod (b_{ii}(m))^{\frac{1}{n}}$$

Together with (8) we obtain the claimed inequality. □

References

[Dan01] F. M. Dannan. Matrix and operator inequalities. *Journal of Inequalities in Pure and Applied Mathematics*, Volume 2, Issue 3, Article 34, 2001.

[Vil03] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.