Real analysis, solutions of the self-study questions.

5.

Show that if f is twice differentiable on $\mathbb{R}$ with $f(0) = 0$ then there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)\left(1 + 2\tan^2(\xi)\right).$$

Hint: use the mean value theorem twice, to $g(x) \coloneqq f(x)\cos x$ *and then to* $h(x) \coloneqq g'(x)/\cos^2 x$.

Solution:

Since $g(-\pi/2) = g(0) = g(\pi/2) = 0$ the mean value theorem gives $\xi_1 \in (-\pi/2, 0)$, $\xi_2 \in (0, \pi/2)$ such that

$$g'(\xi_1) = g'(\xi_2) = 0.$$

Hence also $h(\xi_1) = h(\xi_2) = 0$ and so another application of the mean value theorem gives $\xi \in (\xi_1, \xi_2) \subset (-\pi/2, \pi/2)$ such that

$$\begin{aligned}
0 = h'(\xi) &= \frac{g''(\xi)\cos^2\xi + 2\cos\xi\sin\xi g'(\xi)}{\cos^4\xi} \\
&= \frac{(f''(\xi)\cos\xi - 2f'(\xi)\sin\xi - f(\xi)\cos\xi)\cos\xi + 2\sin\xi(f'(\xi)\cos\xi - f(\xi)\sin\xi)}{\cos^3\xi} \\
&= \frac{1}{\cos\xi}\left(f''(\xi) - f'(\xi)\left(1 - \tan^2\xi\right)\right).
\end{aligned}$$

6.

Let $f \in \mathbb{R} \to (0, \infty)$ be continuously differentiable. Prove that there exists $\xi \in (0,1)$ such that

$$\mathrm{e}^{f'(\xi)} f(0)^{f(\xi)} = f(1)^{f(\xi)}.$$

Solution:

We need to find $\xi \in (0,1)$ such that

$$\begin{aligned}
\left(\frac{f(1)}{f(0)}\right)^{f(\xi)} = \mathrm{e}^{f'(\xi)} &\Leftrightarrow f(\xi)\left(\log f(1) - \log f(0)\right) = f'(\xi) \\
&\Leftrightarrow \frac{f'(\xi)}{f(\xi)} = \log f(1) - \log f(0) \\
&\Leftrightarrow g'(\xi) = g(1) - g(0),
\end{aligned}$$

where $g(x) \coloneqq \log f(x)$. The claim follows from the mean value theorem.

7.

Find all C^1 functions $f\colon [0,1] \to (0, \infty)$ such that

$$\frac{f(1)}{f(0)} = \mathrm{e} \qquad \text{and} \quad \int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 \left(f'(x)\right)^2 \mathrm{d}x \le 2.$$

Hint: $(x + 1/x)^2 = x^2 + 1/x^2 + 2$.

Solution:

$$\begin{aligned}
0 &\le \int_0^1 \left(f'(x) - \frac{1}{f(x)}\right)^2 \mathrm{d}x \\
&= \underbrace{\int_0^1 f'(x)^2 \mathrm{d}x + \int_0^1 \frac{\mathrm{d}x}{f(x)^2}}_{\le 2} - 2 \underbrace{\int_0^1 \frac{f'(x)}{f(x)} \mathrm{d}x}_{=\log f(1) - \log f(0)} \\
&\le 2 - 2 \underbrace{\log \frac{f(1)}{f(0)}}_{=\log \mathrm{e} = 1} = 0.
\end{aligned}$$

Hence, by continuity of f and f', for all x

$$\begin{aligned}
& f'(x) - \frac{1}{f(x)} = 0 \\
\Leftarrow & f'(x) f(x) = 1 \\
\Leftrightarrow & \left(f(x)^2\right)' = 2 \\
\Leftrightarrow & f(x)^2 = 2x + C \\
\Leftarrow & f(x) = \sqrt{2x + C}.
\end{aligned}$$

From the condition $f(1)/f(0) = \mathrm{e}$ we get $C = 2/(\mathrm{e}^2 - 1)$.

8.

Does there exist an injective function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) - f(x)^2 \ge \frac{1}{4}?$$

Hint: when $x^2 = x$?

Solution:

Note $x^2 = x$ iff $x = 0$ or $x = 1$. For each of the two values of x we obtain

$$0 \ge f(x)^2 - f(x) + \frac{1}{4} = \left(f(x) - \frac{1}{2}\right)^2$$

Hence $f(1) = f(0) = 1/2$. No.

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