Handout: Combinatorics

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1 Counting in two ways

1. Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.
Sketch: There are $\binom{200}{2} = 19900$ pairs of contestants. Each problem was missed by at most 80 students. Hence, each problem can be missed by at most $\binom{80}{2} = 3160$ pairs of contestants.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either PASS or FAIL. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $\frac{k}{a} \geq \frac{b-1}{2b}$
Sketch: Consider an $a \times b$ matrix where columns represent the contestants and the rows represents the judges with the corresponding entry to be 1 (resp. 0) when the judge rates the corresponding contestant PASS (resp. FAIL). Say a pair of entries in the same column are *good* if they are equal. Thus the number of good pairs in any two rows is at most k which implies the total number of good pairs in the matrix is at most $k\binom{b}{2}$ In any column, if there are i zeroes, then the total number of good pairs is $\binom{i}{2} + \binom{b-i}{2}$. Write $b = 2m+1$ (since b is odd), then show that $\binom{i}{2} + \binom{b-i}{2} \geq m^2$ which implies the total number of good pairs is at least $am^2 = \frac{a(b-1)^2}{4}$

3. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$-th of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.
Sketch: First show the easier statement that at least 1 contestant solved exactly 5 problems. Then obtain a contradiction to the statement: "exactly 1 contestant solved exactly 5 problems".

2 Probabilistic method

1. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $\frac{n}{2017}$ persons in S have exactly two friends in S.
Sketch: Choose the set S randomly such that each people is selected with probability p (we will fix p later), independently from the others. The probability that a certain person is selected for S and knows exactly two members of S is $q = \binom{1000}{2} \cdot p^3(1-p)^{998}$. Choose p such that $q > \frac{1}{2017}$. Then $\mathbb{E}(|S|) = nq > \frac{n}{2017}$, which implies there is a choice S^* such that $S^* > \frac{n}{2017}$

2. Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. Prove that for every real number q, we have $\sum_P q^{a(P)}(1-q)^{b(P)} = 1$
Sketch: Randomly color the points in black and white, with a point receiving black with probability $0 < q < 1$. For each convex polygon P, let E_P be the event that all vertices on the perimeter of P are black, and all vertices in Ps exterior are white. These events are mutually exclusive, so the LHS is the probability that some E_P holds. But there is always some E_P that always holds: consider the convex hull of all of the black points.

3. **Homework**: A group of 1600 delegates have formed 16000 committees of 80 persons each. Prove that one can find two committees having at least four common members.

3 Pigeonhole Principle (Homework)

1. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26, prove that M contains at least one subset of four distinct elements, whose product is the fourth power of an integer.
Hint: There are only 9 primes smaller than 26.

2. Let $n \geq 2$ be an integer and X be a set of n elements. Let $A_1, A_2, \ldots, A_{101}$ be subsets of X such that the union of any 50 of them has more than $\frac{50n}{51}$ elements. Show that there are three of the A_j's such that the intersection of any two is not empty.
   Hint: Show (via contradiction) that there are at least 51 of these subsets with more than $\frac{n}{51}$ elements each

3. Show that among any 4^{n-1} people, there are either some n of them who mutually know each other, or some n who mutually do not know each other.
   Hint: Show inductively that any large enough graph has either a clique of size n or an independent set of size n.