Linear algebra

1. Express the determinant of the following matrix as a product of linear polynomials.
$$\begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}$$

2. Let A and B be two $n \times n$ matrices. Show that if $A + B = AB$, then $AB = BA$.

3. Do they exist two $n \times n$ matrices A and B such that $AB - BA$ is the unit matrix.

4. Let A and B be two $n \times n$ matrices such that the rank of $AB - BA$ is one. Show that $(AB - BA)^2 = 0$.

5. Let A be an $n \times n$ matrix such that $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \dots, n$. Show that A is regular.

6. Let $X_1, \dots, X_k$ be subsets of $\{1, \dots, n\}$ such that the size of each set X_i is odd and the size of the interesection of any two sets is even. Show that $k \leq n$.

7. (HW) Express the determinant of the following matrix as a product of linear polynomials.
$$\begin{pmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{pmatrix}$$

8. (HW) Do they exist two distinct $n \times n$ matrices A and B such that $A^3 = B^3$, $A^2 B = B^2 A$ and $A^2 + B^2$ is invertible?