Problems from class with detailed solutions

Problem 1 We say that a real $c \in (0,1]$ is a *chord* of a function $f : [0,1] \to \mathbb{R}$ if there is $x \in [0, 1-c]$ with $f(x+c) = f(x)$. Call $c \in (0,1]$ a *universal chord* if it is a chord of every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$. Which reals are universal chords?

Solution: Answer: $c \in (0,1]$ is a universal chord if and only $c = 1/n$ for some integer $n \geq 1$.

First, let us show that $1/n$ for any integer $n \geq 1$ is a universal chord. Take any function f as in the definition. Consider the following telescopic sum:

$$\sum_{i=1}^{n} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right) = f(1) - f(0) = 0.$$

Suppose that no summand is 0 for otherwise we are done. Then there are two summands with different signs. We see that the continuous function $g(x) = f(x+c) - f(x)$ defined on $[0, 1-c]$ changes sign, so it must be 0 at some point by the Intermediate Value Theorem.

Conversely, suppose that $c \in (0,1]$ is not a reciprocal of an integer. Let $g(x) = |\sin(\frac{\pi x}{c})|$ for $x \in \mathbb{R}$. This is a continuous c-periodic function $\mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and $g(1) \neq 0$. Consider $f(x) := g(x) - xg(1)$. It is a continuous function with $f(0) = f(1) = 0$. Furthermore, for every $x \in [0, 1-c]$ we have

$$f(x+c) - f(x) = g(x+c) - (x+c)g(1) - g(x) + xg(1) = -cg(1) \neq 0,$$

that is, f shows that c is not a universal chord, as claimed. ∎

Problem 2 Construct a function $f : [0,1] \to [0,1]$ such that $f(\,(a,b)\,) = [0,1]$ for every $0 \leq a < b \leq 1$.

Solution: We write all reals in binary, agreeing that in the ambiguous cases like

$$0.b_1 \dots b_k 1000 \dots = 0.b_1 \dots b_k 01111 \dots,$$

we always use the latter representation.

On input $x = 0.b_1 b_2 \dots$ written in binary, if there is no k such that $b_{2k} = b_{2k+2} = b_{2k+4} = \dots = 0$, let $f(x) = 0$. Otherwise, take smallest such k and output $0.b_{2k+1} b_{2k+3} \dots$

Take arbitrary $0 \leq a < b \leq 1$ and $y \in [0,1]$. We have to show that there is $x \in (a,b)$ with $f(x) = y$. Write $(a+b)/2$ in binary: $(a+b)/2 = 0.c_1 c_2 c_3 \dots$. Take an even integer m such that $2^{-m} < (b-a)/4$; then any real that begins as $0.c_1 \dots c_m \dots$ is necessarily in (a,b). Write $y = 0.y_1 y_2 \dots$ in binary. Let $x = 0.c_1 \dots c_m 1111 \dots$ if $y = 0$ and $x = 0.c_1 \dots c_m y_1 0 y_2 0 y_3 0 \dots$. It is easy to check that x has the required properties. ∎

Homework problems

These problems give extra practice for the ideas/concepts discuss in the class. They are not part of selection for Warwick's IMC team but you are welcome to hand in your solutions on the next class (or come to my office hours to discuss them).

Problem 3 Call a real $c \in (0,1)$ a *semi-universal chord* if, for every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$, c or $1 - c$ is a chord of f. Which reals in $(0,1)$ are semi-universal chords?

Problem 4 Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions, differentiable on (a,b). Assume in addition that g and g' are nowhere zero on (a,b) and that $f(a)/g(a) = f(b)/g(b)$. Prove that there exists $c \in (a,b)$ such that

$$\frac{f(c)}{g(c)} = \frac{f'(c)}{g'(c)}.$$

Problem 5 Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Prove that there exists $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$