Practice problems

Problem 12 Let $SL_2(\mathbb{Z})$ consists of 2×2-matrices with integer entries and determinant 1.

(i) Show that $SL_2(\mathbb{Z})$ is a group under matrix multiplication.

(ii) Prove that, for every $A \in SL_2(\mathbb{Z})$, the map $x \mapsto Ax$ is bijective on $\mathbb{Z}^2$.

(iii) Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be a non-zero vector in $\mathbb{Z}^2$ such that the straight line segment connecting y to the origin has no points from $\mathbb{Z}^2$ in its interior. Show that there is $A \in SL_2(\mathbb{Z})$ with $Ay = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Problem 13 Suppose that each of the vertices of a triangle ABC belongs to $\mathbb{Z}^2$ and that there is exactly one point P from $\mathbb{Z}^2$ in the interior of the triangle. Let E be the point of intersection of the lines AP and BC. Determine the largest possible value for the ratio of the segments $|AP|/|PE|$.