PROBLEM SOLVING 6TH MARCH

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1. Problems

1. Does there exist
   (a) finitely many (at least 2)
   (b) countably many
   non-empty disjoint closed sets whose union is the interval $[0, 1]$?

2. Let f be a function
   (a) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
   (b) $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$
   such that its restriction to any horizontal or vertical line is a polynomial. Does it follow that f is a polynomial in two variables?

3. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Assume that for every $x > 0$ the sequence $f(nx)$ has a limit. Does it follow that
$$\lim_{x\to\infty} f(x)$$
exists?

4. (If we use a standard numeral system, like the decimal or binary, there are numbers that can be written in two different ways, for example, $10 = 9.999999\ldots$ in base 10, or $0.1 = 0.011111\ldots$ in base 2. This question concerns a bit more general numeral systems.)
   Are there a real number b with $|b| > 1$ and a finite set $D \subset \mathbb{R}$ with $0 \in D$ such that every real number x can be uniquely written in the form
$$x = \sum_{n=-\infty}^{N} d_n b^n$$
where N is an integer, $\forall n\ d_n \in D$, and $d_N \neq 0$?

5. Characterise those functions $f : \mathbb{R} \to \mathbb{R}$ that have the property that for any sequence (a_n),
$$\sum_{n=1}^{\infty} |a_n| < \infty \iff \sum_{n=1}^{\infty} |f(a_n)| < \infty.$$

6. Characterise those functions $f : \mathbb{R} \to \mathbb{R}$ that have the property that for any sequence (a_n),
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} f(a_n) \text{ converges}.$$

In the class we covered (parts of the) solutions of Problem 1, Problem 2 and Baire's category theorem.

Hint for problem 3: use Baire's category theorem.

For Problem 5 and 6 only first-year analysis is required.

If you have solutions or possible solutions, write me (A.Mathe at warwick.ac.uk), and we can fix a time to meet.

Written solutions may be released later.

In the next sections I state Baire's category theorem again and write a solution for problem 1 and 2.

2. Baire's category theorem

Theorem 2.1 (Baire's category theorem (simple form))**.** *If $\mathbb{R}$ (or $[0,1]$) if a union of countably many closed sets, then one of these contains a non-trivial interval.*

In $\mathbb{R}^n$, or in a complete metric space, the theorem states that if the union of countably many closed sets contain an open ball (or the whole $\mathbb{R}^n$ or the whole complete metric space), then one of your closed sets already contains a (non-empty) open ball.

One can also state the theorem by talking about the complements of these closed sets: In $\mathbb{R}$, for example, it says that the intersection of countably many dense open sets is non-empty. (A set is dense if it has a point in every (non-empty) open interval.)

Proof of Baire's category theorem in $\mathbb{R}$. (I will assume that we have countably infinitely many sets, indexed by positive integers, but the same (or shorter) proof works for a finitely many sets as well.) Assume that $\mathbb{R} = \cup_{n=1}^{\infty} A_n$, the sets A_n are closed, but none of them contains an interval.

A_1 does not contain an interval, so there is $x \notin A_1$. Since A_1 is closed, there is an $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon]$ is disjoint from A_1. Let this interval be $[a_1, b_1]$. We know that A_2 does not contain this interval, so there is $x_2 \in [a_1, b_1]$ such that $x_2 \notin A_2$. Since A_2 is closed, we can similarly choose a non-trivial interval $[a_3, b_3]$ that is inside $[a_2, b_2]$ but disjoint from A_2.

And so on. We obtain a nested sequence of closed (non-trivial) intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots,$$

such that $[a_k, b_k]$ is disjoint from A_k. The intersection

$$\bigcap_n [a_n, b_n]$$

is non-empty, as it contains the interval

$$[\sup a_n, \inf b_n].$$

This "interval" often consists of a single point of course (if the lengths of the intervals tend to zero). Since any point in this intersection is disjoint from all the sets A_k, we obtained a contradiction. □

3. Solution of Problem 1

The answer is negative to both questions. The proof does not use Baire's category theorem but it is very similar to its proof.

As many of you have pointed out, $[0,1]$ is not the union of two disjoint closed sets, as it is connected. To see this directly, assume that $[0,1] = A \cup B$, where A and B are disjoint non-empty closed sets. One of the sets does not contain 0, let this be B. Let b be the infimum of B. Since B is closed, $b \in B$ and thus $b > 0$. Since $A \cup B = [0,1] \supset [0,b)$, we must have $A \supset [0,b)$. As A is closed, we have $b \in A$ which contradicts to A and B being disjoint.

If we had

$$[0,1] = \cup_{i=1}^{n} A_i$$

where the sets A_i are disjoint, non-empty and closed, then $[0,1]$ would be the union of two disjoint non-empty closed sets: A_1 and

$$\cup_{i=2}^{n} A_i.$$

So it is left to prove that $[0,1]$ is not a union of countably many disjoint closed sets A_i. Assume it is.

We may assume that $0 \in A_1$ and we have $1 \in A_2$ or perhaps $1 \in A_1$ as well. In either case, $A_1 \cup A_2$ is a closed set in $[0,1]$, which contains 0 and 1 but not everything. Then $[0,1] \setminus (A_1 \cup A_2)$ is a non-empty open set. Choose a (maximal) open interval (a_2, b_2) inside this open set, that is, the points a_2 and b_2 should be in $A_1 \cup A_2$.

Now consider A_3. If $A_3 \cap (a_2, b_2) = \emptyset$, then let $a_3 = a_2$ and $b_3 = b_2$. On the other hand, if $A_3 \cap (a_2, b_2) \neq \emptyset$, then we do one of the following options. We can let $a_3 = a_2$, and let $b_3 = \inf A_3 \cap (a_2, b_2)$; OR we let $b_3 = b_2$ and let $a_3 = \sup A_3 \cap (a_2, b_2)$. In either case we have that the points a_3, b_3 are in $A_1 \cup A_2 \cup A_3$, but the open interval (a_3, b_3) is disjoint from this set.

We repeat the same step. If (a_k, b_k) is given such that it is disjoint from $A_1 \cup \dots A_k$ but its endpoints are in this set, then consider A_k. If $A_k \cap (a_k, b_k) = \emptyset$, then let $a_{k+1} = a_k$ and $b_{k+1} = b_k$. If the intersection is non-empty, then we choose between two options: either let $a_{k+1} = a_k$ and $b_{k+1} = \inf A_{k+1} \cap (a_k, b_k)$ OR we let $b_{k+1} = b_k$ and let $a_{k+1} = \sup A_{k+1} \cap (a_k, b_k)$. In either case we have that the open interval (a_{k+1}, b_{k+1}) is disjoint from $A_1 \cup \dots \cup A_{k+1}$ but its endpoints are in this set.

We obtain a nested sequence of open intervals

$$(a_1, b_1) \supset (a_2, b_2) \supset \cdots .$$

For the nested sequence of closed intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots ,$$

we have that their intersection is non-empty, as it contains the closed interval

$$[\sup a_k, \inf b_k]$$

which is, of course, consists of only a single point in case the length of these intervals tend to zero. The intersection of a nested sequence of open intervals can be empty, but only if one of the endpoints is eventually constant (that is, for example, $a_k = a_{k+1} = a_{k+2} = \dots$ for some k) and the length of the intervals go to zero. We had some freedom in the construction above: it is easy to see that we can make sure that the intersection of these open intervals is still non-empty. Since this point is in $[0, 1]$ but not in $\cup_{n=1}^{\infty} A_n$, we obtain a contradiction.

4. Solution of Problem 2

For $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the answer is positive, for $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, the answer is negative.

Lemma 4.1 (Interpolation)**.** *Let $x_1, \dots, x_d$ be d distinct points in $\mathbb{R}$ and let $y_1, \dots, y_d \in \mathbb{R}$ be arbitrary. Then there is a unique polynomial P of degree at most $d-1$ such that*

$$y_i = P(x_i)$$

for every $i = 1, \dots, d$.

In fact, this polynomial is

$$P(x) = \sum_{n=1}^{d} y_n \cdot \frac{\prod_{i \in \{1,\dots,d\} \setminus \{n\}} (x - x_i)}{\prod_{i \in \{1,\dots,d\} \setminus \{n\}} (x_n - x_i)}.$$

Proof. Notice that

$$\frac{\prod_{i \in \{1,\dots,d\} \setminus \{n\}} (x - x_i)}{\prod_{i \in \{1,\dots,d\} \setminus \{n\}} (x_n - x_i)}$$

is a polynomial of degree $d-1$ that takes value 1 at $x = x_n$ and takes value 0 at all the other points $x = x_i$. So P is indeed the required polynomial, and its degree is at most $d-1$.

If there is another polynomial Q with the same property as P, then $P - Q$ would be a polynomial of degree $d-1$ that has at lest d zeros, so $P - Q = 0$. □

Remark 4.2. Notice that P in the lemma is linear in y_n, if we treat them as variables.

Now assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that on vertical and horizontal lines it is a polynomial. Let

$$E_d = \{x \in \mathbb{R} \, : \, y \mapsto f(x, y) \text{ is a polynomial of degree at most d}\}.$$

Then

$$\mathbb{R} = \bigcup_{d=0}^{\infty} E_d$$

and $E_0 \subset E_1 \subset E_2 \subset \cdots$. Then $\mathbb{R}$ is also the union of the closures of the sets E_d, so there is d such that

$$\overline{E_d}$$

contains a non-trivial interval. Let this be I.

Consider $d+1$ arbitrary horizontal lines, for example, those with y coordinates $0, 1, \dots, d$. On each of these, f is a polynomial, so there exist polynomials P_k such that

$$f(x, k) = P_k(x) \qquad (k = 0, 1, \dots, d).$$

Fix any $x \in E_d$. Then $y \mapsto f(x,y)$ is a polynomial of degree at most d and its value at k is $P_k(x)$ $(k = 0, 1, \ldots, d)$. By the interpolation lemma above, we see that we must have, for every $y \in \mathbb{R}$,

$$f(x,y) = \sum_{k=0}^{d} P_k(x) \cdot \frac{\prod_{i \in \{0,\ldots,d\} \setminus \{k\}} (y-i)}{\prod_{i \in \{0,\ldots,d\} \setminus \{k\}} (k-i)}.$$

Let $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$Q(x,y) = \sum_{k=0}^{d} P_k(x) \cdot \frac{\prod_{i \in \{0,\ldots,d\} \setminus \{k\}} (y-i)}{\prod_{i \in \{0,\ldots,d\} \setminus \{k\}} (k-i)},$$

this is a polynomial in x and y. We already know that f and Q agree on $E_d \times \mathbb{R}$.

For every $c \in \mathbb{R}$, $x \mapsto f(x,c)$ is a polynomial, so continuous, and agrees with another continuous function $x \mapsto Q(x,c)$ on the set E_d. So they must agree on $\overline{E_d}$ as well. Hence, they agree on the interval I. If two polynomials agree on an interval, then they are the same, so we must have $f(x,c) = Q(x,c)$ for every $x \in \mathbb{R}$ (for every c). So we obtained that f and Q agree everywhere in the plane, so f is indeed a polynomial.

For the second part of the problem we construct a function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ that is not a polynomial in two variables, but it is a polynomial on every horizonal and vertical line.

Enumerate rationals in a sequence $\{r_1, r_2, \ldots\}$. Consider the horizontal and vertical lines $\{(x, r_1) : x \in \mathbb{R}\}$, $\{(r_1, y) : y \in \mathbb{R}\}$, $\{(x, r_2) : x \in \mathbb{R}\}$, $\{(r_2, y) : y \in \mathbb{R}\}$, ... We will fix the values of f recursively on each of these lines, in this order.

Fix f on the first line (which is horizontal) to be any polynomial of degree at least 1. Then fix f on the second line (which is vertical) to be any polynomial of degree at least 2 (keeping in mind that f was already defined at the intersection of the first and second line). And so on, we fix f on the kth line to be any polynomial of degree at least k. We can do this, because the value of f of that line is only fixed at finitely many points (that are the intersections with previously considered lines). We can also easily ensure that the degree of the polynomial on the new line is as large as we wish (check this, and use the "interpolation" lemma if needed).

At the end of this process we defined $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, which is a polynomial on all horizontal and vertical lines. It cannot be a polynomial in two variables, because then it would have a degree d, and then its restrictions to lines would have a degree at most d. However, we made sure we have arbitrarily large degrees on the lines.