IMC Selection Test 1

- *Write your full name and email on the first sheet*
- *Time:* ***2 hours 50 minutes***
- *Books, notes and calculators* ***are not allowed***

Problem 1 Let $f : \mathbb{N} \to \mathbb{N}$ be the function satisfying $f(1) = 1$, $f(2n) = f(n)$, and $f(2n+1) = f(2n) + 1$, for all positive integers n. Find the maximum value of $f(n)$ when $1 \leq n \leq 2020$.

Problem 2 Let

$$f(x) := \begin{cases} 0, & x = \pi/2 + k\pi \text{ for } k \in \mathbb{Z}, \\ \frac{1}{2+(\tan x)^2}, & \text{otherwise.} \end{cases}$$

For which $a \in \mathbb{R}$ is the function $g(x) := f(x) + f(ax)$, $x \in \mathbb{R}$, periodic?

Problem 3 Let G be a group written multiplicatively with the identity e. Let $\phi : G \to G$ a function such that

$$\phi(x_1)\phi(x_2)\phi(x_3) = \phi(y_1)\phi(y_2)\phi(y_3)$$

whenever $x_1x_2x_3 = e = y_1y_2y_3$. Prove that there exists an element $a \in G$ such that $\psi(x) := a\phi(x)$, $x \in G$, is a homomorphism of the group G (i.e., $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

Problem 4 Let $D := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk in the complex plane. Which monic polynomials, $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ with complex coefficients satisfy $f(D) \subseteq D$ (that is, satisfy $|p(z)| \leq 1$ for all $z \in D$)?