IMC Selection Test 2

- *Write your full name and email on the first sheet*
- *Time:* ***2 hours 50 minutes***
- *Books, notes and calculators* ***are not allowed***

Problem 1 Let $\lambda_1, \ldots, \lambda_n$ be non-zero reals and let $A = (a_{ij})_{i,j=1}^n$ be the $n \times n$-matrix with entries $a_{ij} = \lambda_i/\lambda_j$ for $i, j \in \{1, \ldots, n\}$. Find the eigenvalues of A.

Problem 2 Let a set X of size 2^n be partitioned into some subsets $A_1, \ldots, A_m$. We can repeat the following operation: for some sets A_i and A_j with $|A_i| \geq |A_j|$, move $|A_j|$ elements from A_i to A_j. (Thus the size of A_j doubles.) Prove that starting from any initial partition we can make one set to be equal to the whole set X.

Problem 3 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. Let $n, a, b, c \in \mathbb{N}$ satisfy $n = a^2 + b^2 + c^2$. Prove that every natural power of n is also a sum of three non-zero squares, that is, for every $k \in \mathbb{N}$ there are $A, B, C \in \mathbb{N}$ with $n^k = A^2 + B^2 + C^2$.

Problem 4 Find all real solutions to the system of equations

$$\begin{aligned} 2x + x^2 y &= y, \\ 2y + y^2 z &= z, \\ 2z + z^2 x &= x. \end{aligned}$$